

MASS-LIKE INVARIANTS FOR ASYMPTOTICALLY HYPERBOLIC METRICS

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ABSTRACT. In this article, we classify the set of asymptotic mass-like invariants for asymptotically hyperbolic metrics. It turns out that the standard mass is just one example (but probably the most important one) among the two families of invariants we find. These invariants are attached to finite-dimensional representations of the group of isometries of hyperbolic space. They are then described in terms of wave harmonic polynomials and polynomial solutions to the linearized Einstein equations in Minkowski space.

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1. INTRODUCTION

1.1. Background. Asymptotically hyperbolic manifolds are non-compact Riemannian manifolds having an end on which the geometry approaches that of hyperbolic space. Such manifolds have attracted a lot of attention for the last decades, in theoretical physics as well as in geometry.

On the one hand, they appear naturally in general relativity, in the description of isolated gravitational systems in a universe with negative cosmological constant (see for example [3]), modeled on Anti-de Sitter spacetime. One can also see them arising as asymptotically umbilical hypersurfaces in asymptotically Minkowski spacetimes. However, the point of view of general relativity also involves the second fundamental form of the embedded hypersurface. In a forthcoming paper we will generalize the results from the Riemannian setting treated in this paper to the setting of hypersurfaces with second fundamental form.

On the other hand, asymptotically hyperbolic manifolds have their own geometric interest. An important example is the study of conformally compact Riemannian manifolds, in particular the Fefferman-Graham theory of the ambient metric: given a sufficient amount of data on a closed $n - 1$ -dimensional manifold (in particular prescribing a conformal class on it), it occurs then as the conformal boundary of a unique asymptotically hyperbolic, Einstein n -dimensional manifold. See [19] for a review of the theory.

For general asymptotically hyperbolic (non-necessarily Einstein) manifolds, one needs a *chart at infinity*, in other words a coordinate system through which one can measure the decay rate of the metric towards the hyperbolic metric. Note that in the conformally compact case, this can be given by a defining function of the conformal boundary. Some interesting results can be found when the decay rate as read in a chart at infinity is high enough and if one moreover assumes some positivity condition on the curvature tensor. The first achievements in this directions were *scalar curvature rigidity* results, for which the scalar curvature is required to be greater than or equal to the one of the hyperbolic space of same dimension. Under completeness and strong enough decay assumptions as well as a further topological (spin) assumption, the manifold has to be isometric to the hyperbolic space, see [2, 37].

Motivated by the study of isolated systems in general relativity, properties of metrics with weaker decay assumptions were investigated. The difference between the metric g and the hyperbolic metric b is quantified by the so-called *mass vector* $\mathbf{p}_g \in \mathbb{R}^{n+1}$. This notion is reminiscent of the *ADM mass* for an asymptotically Euclidean manifold, and “positive mass theorems”, with their rigidity conclusions in the case of vanishing mass, have been obtained [1, 13, 43]. See also the recent work [41] and [18].

As for asymptotically Euclidean manifolds, one of the difficulties when studying the mass is its apparent dependence on the chart at infinity. This problem appears in both the approach by Wang [43] and by Chruściel and Herzlich [13]. In the latter work it is however established that a scalar quantity formed from the mass vector, the mass, does not depend on a given chart at infinity provided it satisfies the appropriate asymptotic conditions.

The fundamental result at the origin of this fact was derived first by Chruściel and Nagy in [14, Theorem 3.3]. It states in particular that the transition diffeomorphism $\Psi = \varphi_2 \circ \varphi_1^{-1}$ between any two charts at infinity φ_1 and φ_2 is composed of a principal part A which is an isometry of the hyperbolic space, and a correction part that decays to zero on approach to infinity. The decay of the correction part is in fact sufficiently fast so that it does not alter the expression of the components of the *mass vector*, themselves computed at infinity. The transition diffeomorphism Ψ is for this reason called an *asymptotic isometry*. We denote by π the projection map $\Psi \mapsto A$ from the group of asymptotic isometries to finite-dimensional subgroup of hyperbolic isometries.

Wang and Chruściel-Herzlich then established that the mass vector \mathbf{p}_g of any asymptotically hyperbolic metric of suitable decay enjoys the equivariant property

$$\mathbf{p}_{\Psi * g} = \pi(\Psi) \mathbf{p}_g,$$

where $\pi(\Psi)$ acts naturally as an element of the Lorentz group $O(n, 1)$ on \mathbb{R}^{n+1} . It follows in particular that the norm in the Minkowski metric η , $\eta(\mathbf{p}_g, \mathbf{p}_g)$, is invariant under the action of asymptotic isometries.

The essential difference compared to the asymptotically Euclidean case comes rather from the nature of the mass. Whereas the mass is encoded in a single number (the ADM mass) in the asymptotically Euclidean case, it is instead an $(n + 1)$ -vector in the hyperbolic case, which transforms equivariantly under the action of hyperbolic isometries. This vector also encodes the *center of mass*, see [11]. It is only when considering the Minkowski quadratic form evaluated on this vector (invariant under the action of hyperbolic isometries) that one gets a number independent of the choice of the chart at infinity. We therefore call it an *linear mass at infinity* of the asymptotically hyperbolic manifold.

This difference in nature between these masses can be explained by the existence of a much bigger conformal infinity for the hyperbolic space (a codimension 1 sphere at infinity) than for the Euclidean space (a single

point at infinity). With the large conformal infinity of hyperbolic space one could speculate that asymptotically hyperbolic manifolds should have other linear masses at infinity, and one may look for the full list of them.

This investigation is also motivated by the study of asymptotically hyperbolic *Poincaré-Einstein* metrics, where asymptotic invariants do appear, see for example [19] and references therein. However, the renormalized volume introduced by Graham in [24] does not fall into our classification since the asymptotic structure is different and the invariant depends on the whole geometry of the manifold, not only on its asymptotics.

1.2. Statement of results. The goal of this paper is to make a step towards the classification of *linear masses at infinity* for asymptotically hyperbolic manifolds whose Riemannian metrics decay towards the standard metric of the hyperbolic space at *any* specified rate.

Motivated by the example of the mass vector, we aim at finding quantities $\Phi(g)$, defined for asymptotically hyperbolic metrics g in a neighborhood of the infinity of the hyperbolic space \mathbb{H}^n , which live in a finite-dimensional representation V of the group of hyperbolic isometries.

Roughly speaking, such a quantity is a *linear mass at infinity* if the map $\Phi : g \mapsto \Phi(g) \in V$ is linear in $g - b$, where b denotes the hyperbolic metric, and satisfies the equivariance property

$$\Phi(\Psi_*g) = \pi(\Psi) \cdot \Phi(g) \quad (1)$$

under the action of the group of asymptotic isometries.

Our results take into account the decay rate of the metric. We will denote by G_k the space of asymptotically hyperbolic metrics g that decay to order k towards b at infinity, meaning that the norm $|g - b|_b$ satisfies

$$|g - b|_b = O\left(e^{-kd(x)}\right),$$

where $d(x) = \text{dist}_b(x, x_0)$ is the b -distance between x and a given point $x_0 \in \mathbb{H}^n$. A more precise definition will be given in Section 2.

We now state a loose version of our classification results obtained in Theorem 4.2 and Theorem 4.4 for $n = 3$, and in Theorem 4.5 for all $n \geq 4$.

Theorem 1.1. *There exists two families of linear masses*

$$\{\Phi_c^{(k)} : G_k \rightarrow V_k\}_{k \geq n-1} \text{ and } \{\Phi_w^{(k)} : G_k \rightarrow W_k\}_{k \geq n+1}$$

indexed by integers k , where V_k and W_k are irreducible, finite-dimensional representations of the group $O_\uparrow(n, 1)$ of hyperbolic isometries.

Moreover, given any finite-dimensional representation V and a positive number k , the map $\Phi : G_k \rightarrow V$ is a non-trivial linear mass at infinity if and only if it is a linear combination of $\Phi_c^{(k)}$ and $\Phi_w^{(k)}$. In particular, k is then an integer larger than $n - 1$.

The maps $\Phi_c^{(k)}$ are called *conformal masses*, while the $\Phi_w^{(k)}$ are called *Weyl masses*, for reasons which will become clearer in Section 4.

The classical Wang-Chruściel-Herzlich mass vector is obtained here as the linear mass $\Phi_c^{(n)}$, for which V_n is the standard representation of $O(n, 1)$, that is the $n + 1$ -dimensional Minkowski space.

In Section 4, the maps Φ being classified are defined on the set of *mass-aspect tensors*, that is the set of symmetric $(2, 0)$ -tensors over the unit sphere \mathbb{S}^{n-1} . Such tensors can be thought of as the coefficient of the first non-trivial term of a metric g in G_k . It will be argued in Sections 2, 3 and 5 that there is no loss of generality in looking for intertwining maps Φ defined on mass-aspect tensors instead.

Once our classification is obtained, we will link in Section 6 the linear masses at infinity so found with properties of geometric operators in the spirit of Michel [36]. As is well known for the classical mass, the representation $(V_n)^*$, dual to Minkowski, coincides with the space $\ker P_0^*$, where P_0 is the linearized scalar curvature at operator at $g = b$, and the map $\Phi_c^{(n)}$ coincides with the mass functional given by Chruściel and Herzlich in [13]. An interesting point is that this other way of defining the mass does not assume any a priori asymptotic expansion form of the metric and holds for more general asymptotics.

1.3. Overview of the paper. The paper is structured as follows. In Section 2 the relevant definitions are introduced and followed by the description of the group of *asymptotic isometries*. Its elements are seen as transition maps between two asymptotic charts in which the metric has a given decay rate towards the reference hyperbolic metric b . The description of this group culminates in Theorem 2.11 which shows, in the spirit of [14, Theorem 3.3], that such an asymptotic isometry is essentially an isometry of the hyperbolic space, composed with a diffeomorphism asymptotic to the identity.

Meanwhile, we introduce the subclass of *transverse* germs of metrics among the class of asymptotically hyperbolic germs of metrics with a given decay rate. Those are the (germs of) metrics defined on a neighborhood of infinity of \mathbb{H}^n whose expressions have a normal form in the standard hyperbolic coordinates, like the hyperbolic metric b itself. We show in Proposition 2.7 that any germ of metrics g , asymptotic to b , can be sent to a transverse germ θ_*g of the same decay rate, using an *adjustment diffeomorphism* θ . Such adjustment diffeomorphisms have the property of being asymptotic to the identity at sufficiently high order. We make use of this in Proposition 2.12 to define an action of the group of hyperbolic isometries on such transverse metrics.

The last part of Section 2 is devoted to the definition of linear masses at infinity, for which we prefer to work with the notion of *jets*, obtained as some quotient space from stalks. We show that all the facts holding above for germs (transversality, adjustment, group action) descends naturally to jets. We are then in position to define the notion of *linear mass at infinity* in Definition 2.19 and to make precise sense of (1).

In Sections 3 and 4, we investigate linear masses at infinity for germs of asymptotically hyperbolic metrics. From the Definition 2.19, the intertwining property (1) should also hold for the quotient map at the level of transverse jets of metrics with any given decay rate.

We argue in Section 3 that considering a transverse jet of a metric g is equivalent to considering the first non-trivial term m of the asymptotic expansion of g . This term is called the *mass-aspect tensor* of g . We then find expressions for the action of the group of hyperbolic isometries on mass-aspect tensors. It depends on the decay order at which they appear in the asymptotic expansion of the metric. We also argue in Proposition 3.3 that the classification of linear masses at infinity is equivalent to the classification of maps $\Phi : m \mapsto \Phi(m) \in V$, defined on the space of mass-aspect tensors, and which are intertwining with respect to the action of the group of hyperbolic isometries.

The sequel of Section 3 deals with the intertwining property satisfied by a linear mass at infinity Φ when one descends to the actions by the *Lorentz Lie algebra*. To make computations more tractable, we decide from here to look for all the maps $\Phi : m \mapsto \Phi(m) \in V$ which are intertwining for the Lorentz Lie algebra action.

The following Section 4 is the central section of our paper. It establishes first the classification of all the Lorentz Lie algebra intertwining maps $\Phi : m \mapsto \Phi(m) \in V$, where V is a finite-dimensional, irreducible representation of the Lorentz algebra, and where, again, the action on mass-aspect tensors m depend on the decay order of the metrics in the asymptotic expansion of which they appear.

We separate the $n = 3$ -dimensional case on the one hand, and the higher $n \geq 4$ -dimensional case on the other hand. For both, we classify first the possible finite-dimension representations V of the Lorentz Lie algebra, and we relate the highest weights of each such representation with the decay order of the metrics.

We obtain at the end (Theorems 4.2, 4.4, 4.5) two families of representations, giving two families of intertwining maps, parametrized by the decay rate of the metric (which, again, plays a crucial role in the definition of the action on mass-aspect tensors). One last remaining fact that we check is whether the Lie algebra intertwining maps so obtained are intertwining for the full group of hyperbolic isometries as well. This is indeed the case, thus providing the complete classification of linear masses at infinity.

In Section 5, we give the proof of a technical fact about jets of asymptotically hyperbolic metrics of a given order. It allows us to assert that the setting chosen in Section 3 is sufficient to obtain all the linear masses at infinity.

In Section 6, we show how to recover the linear masses at infinity previously obtained using B. Michel's formalism [36]. We recall first the example of the Chruściel-Herzlich [13] mass which can be assigned to asymptotically hyperbolic metrics with weaker asymptotic decay assumptions.

We show then that the linear masses at infinity previously found actually stem from geometric differential operators, similar to the way the mass stems from the scalar curvature operator. These operators takes their values either in the space of smooth functions, or in the space of sections of symmetric 2-tensors over the hyperbolic space. In both cases, we can decompose these spaces in terms of the representations that already appeared in the classification in Section 4. We then give explicit examples of geometric differential operators which give the linear masses obtained in that Section.

This can be compared with Herzlich's recent study of the asymptotically Euclidean case [29], where asymptotic invariants stemming from a class of admissible curvature operators turn out to be nothing but the ADM mass, up to a constant factor.

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2. PRELIMINARIES AND BASIC DEFINITIONS

In this section we give the basic definitions that will be used throughout the paper.

2.1. Hyperbolic space. The reference Riemannian manifold in the context under consideration is hyperbolic space. Unless stated otherwise we will use the Poincaré ball model where hyperbolic space \mathbb{H}^n is described as the unit ball $B_1(0)$ centered at the origin in \mathbb{R}^n equipped with the metric

$$b := \rho^{-2}\delta, \tag{2}$$

where δ denotes the Euclidean metric on \mathbb{R}^n and the function ρ is defined by

$$\rho(x) := \frac{1 - |x|^2}{2}.$$

There are several models of hyperbolic space, but this one seems most convenient in our context. The interested reader can consult for example [7].

The hyperbolic distance in the ball model is given by

$$\cosh d^b(x, y) = 1 + \frac{|x - y|^2}{2\rho(x)\rho(y)} \quad (3)$$

for $x, y \in B_1(0)$.

Since the isometry group of hyperbolic space will play a prominent role in what follows, we will discuss it briefly now. Let $\mathbb{R}^{n,1}$ denote Minkowski space, that is \mathbb{R}^{n+1} equipped with the quadratic form

$$\eta := -(dX^0)^2 + \sum_{k=1}^n (dX^k)^2.$$

We denote by X^0, X^1, \dots, X^n the standard coordinates on $\mathbb{R}^{n,1}$. Further, we denote by $\partial_0, \partial_1, \dots, \partial_n$ the standard basis of $\mathbb{R}^{n,1}$, which has the dual basis of one-forms dX^0, dX^1, \dots, dX^n . Hyperbolic space can be embedded into Minkowski space as the unit hyperboloid,

$$\mathbb{H}^n = \left\{ (X^0, X^1, \dots, X^n) \in \mathbb{R}^{n,1} \mid -(X^0)^2 + \sum_{k=1}^n (X^k)^2 = -1, X^0 > 0 \right\}.$$

The orthogonal group $O(n, 1)$ consists of linear maps preserving the quadratic form η . It has 4 connected components. Indeed, an element $A \in O(n, 1)$ can have determinant ± 1 and the scalar product $\eta(\partial_0, A\partial_0)$ can be either negative, meaning that ∂_0 and $A\partial_0$ point in the same direction (in which A is said to be future preserving), or positive (and A is then called future reversing). The two components mapping hyperbolic space to itself are the future preserving ones. They form the *orthochronous Lorentz group* denoted by $O_\uparrow(n, 1)$. The connected component of the identity (that is, the future preserving isometries with positive determinant) is called the *restricted Lorentz group* and is denoted by $SO_\uparrow(n, 1)$.

The group $O_\uparrow(n, 1)$ is the group of isometries of hyperbolic space while $SO_\uparrow(n, 1)$ is its subgroup of orientation preserving isometries. This subgroup coincides with the connected component of the identity of $O(n, 1)$, see for example [39, Chapter 5, Section 3.10] for details.

The ball model of hyperbolic space is obtained from the hyperboloid model via a stereographic projection p to the plane $X^0 = 0$ with respect to the point $(-1, 0, \dots, 0)$,

$$p(X^0, X^1, \dots, X^n) := \frac{1}{1 + X^0}(X^1, \dots, X^n).$$

We identify the plane $X^0 = 0$ with \mathbb{R}^n , and we denote the standard coordinates on \mathbb{R}^n by x^1, \dots, x^n . The inverse of p is given by

$$p^{-1}(x^1, \dots, x^n) = \left(\frac{1 + |x|^2}{1 - |x|^2}, \frac{2x^1}{1 - |x|^2}, \dots, \frac{2x^n}{1 - |x|^2} \right),$$

where $|x|^2 = (x^1)^2 + \dots + (x^n)^2$.

If $A \in O_{\uparrow}(n, 1)$ is an isometry of the hyperboloid, we transfer it to an isometry of the ball model acting as $\bar{A} = pAp^{-1}$. In what follows we will mainly restrict ourselves to elements belonging to the subgroup $SO_{\uparrow}(n, 1)$ and consider two particular types of such elements.

- A *rotation* by an angle θ in the $X^i X^j$ -plane ($1 \leq i, j \leq n$) is given by

$$\begin{aligned} R_{ij}^{\theta}(X^0, \dots, X^i, \dots, X^j, \dots, X^n) \\ = (X^0, \dots, \cos(\theta)X^i - \sin(\theta)X^j, \dots, \sin(\theta)X^i + \cos(\theta)X^j, \dots, X^n). \end{aligned}$$

We denote the corresponding infinitesimal generator with a script letter,

$$r_{ij} := \frac{d}{d\theta} R_{ij}^{\theta}|_{\theta=0} = dX^i \partial_j - dX^j \partial_i.$$

Note that rotations commute with p , so $\bar{R} := pRp^{-1}$ reads

$$\begin{aligned} \bar{R}_{ij}^{\theta} &:= pR_{ij}^{\theta}p^{-1}(x^1, \dots, x^i, \dots, x^j, \dots, x^n) \\ &= (x^1, \dots, \cos(\theta)x^i - \sin(\theta)x^j, \dots, \sin(\theta)x^i + \cos(\theta)x^j, \dots, x^n). \end{aligned}$$

The derivative of $pR_{ij}^{\theta}p^{-1}$ with respect to θ at $\theta = 0$ is the rotation vector field \mathfrak{r}_{ij} , where

$$\mathfrak{r}_{ij} := x^i \partial_j - x^j \partial_i. \quad (4)$$

- A *Lorentz boost* in the direction X^i ($1 \leq i \leq n$) with a parameter $s \in \mathbb{R}$ is given by

$$\begin{aligned} A_i^s(X^0, \dots, X^i, \dots, X^n) \\ = (\cosh(s)X^0 + \sinh(s)X^i, \dots, \sinh(s)X^0 + \cosh(s)X^i, \dots, X^n). \end{aligned}$$

The corresponding infinitesimal generator is given by

$$a_i := \frac{d}{ds} A_i^s|_{s=0} = dX^0 \partial_i + dX^i \partial_0.$$

As before, we express the corresponding isometry of the ball model as

$$\begin{aligned} \bar{A}_i^s &:= pA_i^s p^{-1}(x^1, \dots, x^i, \dots, x^n) \\ &= \frac{1}{D} \left(x^1, \dots, \cosh(s)x^i + \sinh(s)\frac{1+|x|^2}{2}, \dots, x^n \right), \end{aligned} \quad (5)$$

where

$$D := \frac{1-|x|^2}{2} + \frac{1+|x|^2}{2} \cosh(s) + x^i \sinh(s).$$

The derivative of \bar{A}_i^s with respect to s at $s = 0$ is the boost vector field \mathfrak{a}_i , where

$$\mathfrak{a}_i := \frac{1+|x|^2}{2} \partial_i - x^i x^a \partial_a. \quad (6)$$

In this article we will use the convention that upper case latin letters denote elements in the Lie group $O_{\uparrow}(n, 1)$, while lower case latin letters will be used for elements in the Lie algebra $\mathfrak{so}(n, 1)$ and fraktur letters will denote the corresponding vector fields on the ball $B_1(0) \subset \mathbb{R}^n$, or the same vector fields restricted to the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. These vector fields are actually tangential when restricted to the sphere since the sphere \mathbb{S}^{n-1} is preserved by elements of $O_{\uparrow}(n, 1)$.

2.2. Asymptotically hyperbolic metrics. We continue by defining asymptotically hyperbolic manifolds. Several definitions exist in the literature, we refer the reader to [22, 28, 32] for an overview. To avoid technical complications we choose here to use the simplest such definition. The relevance of this choice will be discussed in Section 6.

Definition 2.1. Let \overline{M} be a compact manifold of dimension n with boundary $\partial M \simeq \mathbb{S}^{n-1}$. We choose a neighborhood Ω of ∂M in \overline{M} , a diffeomorphism $\Psi : \Omega \rightarrow \overline{B}_1(0) \setminus \overline{B}_{1-\varepsilon}(0)$ from Ω to the standard annulus and a positive integer k . Then a metric g on the interior M of \overline{M} is said to be *asymptotically hyperbolic* of order k with respect to Ψ if the metric

$$\overline{g} := \rho^2 \Psi_* g,$$

which is a priori defined only on $B_1(0) \setminus \overline{B}_{1-\varepsilon}(0)$, extends to a smooth metric on $\overline{B}_1(0) \setminus \overline{B}_{1-\varepsilon}(0)$ such that

$$|\overline{g} - \delta|_{\delta} = O\left(\rho^k\right).$$

The boundary $\partial M \simeq \mathbb{S}^{n-1}$ is called the *sphere at infinity* for the asymptotically hyperbolic metric g .

If the metric g is asymptotically hyperbolic of order k with respect to two diffeomorphisms $\Psi_i : \Omega_i \rightarrow \overline{B}_1(0) \setminus \overline{B}_{1-\varepsilon_i}(0)$ for $i = 1, 2$, then the composition $\Psi_2 \circ \Psi_1^{-1}$ is a diffeomorphism between two neighborhoods of \mathbb{S}^{n-1} in $\overline{B}_1(0)$. Hence we will focus on the study of asymptotically hyperbolic metrics in neighborhoods of the sphere at infinity \mathbb{S}^{n-1} in $\overline{B}_1(0)$. Our results will however apply for any manifold (M, g) which is asymptotically hyperbolic with respect to some diffeomorphism Ψ as in Definition 2.1.

Let $r = r(x)$ denote the distance from the origin in the ball model of \mathbb{H}^n . From Formula (3) we have

$$\rho(x) = \frac{1}{\cosh(r) + 1}.$$

As a consequence, the estimate for the decay of the metric can be rewritten in a more intrinsic way as

$$|g - b|_b = O\left(e^{-kr}\right), \quad (7)$$

and a simple argument using the triangle inequality shows that replacing r by the distance function from any given point in $B_1(0)$ gives an equivalent decay estimate.

Since we are interested in the asymptotic behavior of asymptotically hyperbolic metrics, it is sufficient to restrict our attention to germs of such metrics. An open subset $U \subset B_1(0)$ is called a *neighborhood of infinity* if $B_1(0) \setminus \overline{B_{1-\varepsilon}}(0) \subset U$ for some $\varepsilon > 0$. We denote by \mathcal{N}_∞ the set of neighborhoods of infinity. Given an element $U \in \mathcal{N}_\infty$ we define

$$\tilde{U} := U \cup \mathbb{S}^{n-1},$$

that is \tilde{U} is the set of points in U together with the sphere at infinity.

Definition 2.2. For a positive integer k , we consider the set G_k^0 consisting of pairs (U, g) where U is a neighborhood of infinity and g is a metric on U such that $\bar{g} := \rho^2 g$ extends to a smooth metric on \tilde{U} satisfying

$$|g - b|_b = |\bar{g} - \delta|_\delta = O(\rho^k).$$

The relation \sim is defined by $(U_1, g_1) \sim (U_2, g_2)$ if there is a $U_3 \in \mathcal{N}_\infty$ such that $U_3 \subset U_1 \cap U_2$ and $g_1 \equiv g_2$ on U_3 . This is an equivalence relation on G_k^0 and we define the *stalk at infinity of asymptotically hyperbolic metrics of order k* as $G_k := G_k^0 / \sim$. An element $g \in G_k$ will be called an *asymptotically hyperbolic germ at infinity*.

We will abuse notation, and blur the distinction between an element $g \in G_k$ and the metric in an element $(U, g) \in G_k^0$ representing g . This terminology is modeled on standard terminology of sheaf theory, see for example [10, Chapter 1] or [27, Chapter 2]. Note that the stalk at infinity of asymptotically hyperbolic metrics of order k is an affine space.

We will now introduce the stalk at infinity of asymptotic isometries of an asymptotically hyperbolic metric.

Definition 2.3. Given a positive integer k and $g \in G_k$, we define the set $I^k(g)$ of *asymptotic isometries* of g as the stalk at infinity of diffeomorphisms $\Psi : U \rightarrow V$ where U and V are neighborhoods of infinity such that $\Psi_* g \in G_k$.

In Lemma 2.10 it will be proven that $I^k(g)$ does not depend on the choice of $g \in G_k$. As a corollary we will see that $I^k(g)$ is a group under composition of maps.

We also introduce a particular class of germs of asymptotically hyperbolic metrics. This class will play an important role in what follows.

Definition 2.4. Let g be a germ of an asymptotically hyperbolic metric of order k . We say that g is *transverse* if there exists (U, g) representing g such that

$$g_{ij}x^i = b_{ij}x^i$$

on U , or equivalently,

$$g(P, \cdot) = b(P, \cdot)$$

on U , where $P := x^i \partial_i$ is the position vector field on \mathbb{R}^n . We denote by G_k^T the set of transverse asymptotically hyperbolic germs of order k .

A similar condition appears in the context of asymptotically hyperbolic Einstein metrics through the notion of *geodesic defining functions*, see for example [19, Lemme 2.1.2], and in Wang's approach to defining the mass, see [43].

In this definition, and in the rest of the paper, we apply the convention that an expression with an index appearing both as upper and lower index is summed over the appropriate range for that index.

Note that the transversality condition can be stated in terms of the dual of the metric as

$$g(d|x|^2, \cdot) = b(d|x|^2, \cdot)$$

on U , or equivalently in terms of the covectors dx^i as

$$\bar{g}(d\rho, dx^i) = -x^i \quad (8)$$

on U . In particular,

$$\begin{aligned} g(d|x|^2, d|x|^2) &= b(d|x|^2, d|x|^2), \\ g(d\rho, d\rho) &= b(d\rho, d\rho), \\ \bar{g}(d\rho, d\rho) &= \delta(d\rho, d\rho) = |x|^2, \end{aligned}$$

and

$$|d\rho|_{\bar{g}}^2 = 1 - 2\rho. \quad (9)$$

Next we define diffeomorphisms asymptotic to the identity.

Definition 2.5. Given a positive integer k and a neighborhood of infinity U , we say that a diffeomorphism $\Theta : \tilde{U} \rightarrow \Theta(\tilde{U})$ is asymptotic to the identity of order k if

$$|\Theta(x) - x|_{\delta} = O(\rho^k).$$

We denote by I_0^k the stalk at infinity of diffeomorphisms asymptotic to the identity. The stalk I_0^k is a group under composition.

The next lemma tells us that a diffeomorphism asymptotic to the identity of order $k+1$ is an asymptotic isometry of order k for any germ of metric of order k .

Lemma 2.6. For $g \in G_k$ we have $I_0^{k+1} \subset I^k(g)$.

Proof. The stalk I_0^{k+1} is a group under composition, thus we can check that $\Theta^*g \in G_k$ for all $\Theta \in I_0^{k+1}$ since this is equivalent to $(\Theta^{-1})_*g \in G_k$. In component notation we have

$$\begin{aligned} \rho^2(x)(\Theta^*g)(x)_{ij} - \delta_{ij} &= \rho^2(x)g(\Theta(x))_{kl}\partial_i\Theta^k\partial_j\Theta^l - \delta_{ij} \\ &= \frac{\rho^2(x)}{\rho^2(\Theta(x))}\bar{g}(\Theta(x))_{kl}\partial_i\Theta^k\partial_j\Theta^l - \delta_{ij}. \end{aligned}$$

By a standard trick we prove that the function $\frac{\rho \circ \Theta}{\rho}$ is smooth near \mathbb{S}^{n-1} . Write

$$\begin{aligned}
\Theta^i(x) &= \Theta^i\left(\frac{x}{|x|}\right) - \int_{|x|}^1 \frac{d}{ds} \Theta^i\left(s \frac{x}{|x|}\right) ds \\
&= \frac{x^i}{|x|} - \int_{|x|}^1 \partial_j \Theta^i\left(s \frac{x}{|x|}\right) \frac{x^j}{|x|} ds \\
&= \frac{x^i}{|x|} - (1 - |x|) \int_0^1 \partial_j \Theta^i\left((\lambda + (1 - \lambda)|x|) \frac{x}{|x|}\right) \frac{x^j}{|x|} d\lambda \\
&= \frac{x^i}{|x|} - (1 - |x|) \left(\frac{x^i}{|x|} + \int_0^1 \left[\partial_j \Theta^i\left((\lambda + (1 - \lambda)|x|) \frac{x}{|x|}\right) - \delta_j^i \right] \frac{x^j}{|x|} d\lambda \right) \\
&= x^i - (1 - |x|) \underbrace{\frac{x^j}{|x|} \int_0^1 \left[\partial_j \Theta^i\left(\lambda \frac{x}{|x|} + (1 - \lambda)x\right) - \delta_j^i \right] d\lambda}_{=: E^i(x)},
\end{aligned}$$

where we set $s = \lambda + (1 - \lambda)|x|$. The vector field E^i is smooth near \mathbb{S}^{n-1} and has components satisfying $E^i(x) = O(\rho^k)$. Thus,

$$\begin{aligned}
\frac{\rho \circ \Theta}{\rho}(x) &= \frac{1 - |\Theta(x)|^2}{1 - |x|^2} \\
&= 1 + 2 \frac{(1 - |x|)x^i E^i(x)}{1 - |x|^2} - \frac{(1 - |x|)^2 |E(x)|^2}{1 - |x|^2} \\
&= 1 + 2 \frac{x^i E^i(x)}{1 + |x|} - \frac{1 - |x|}{1 + |x|} |E(x)|^2.
\end{aligned} \tag{10}$$

From this we conclude that $\frac{\rho \circ \Theta}{\rho}$ is smooth and satisfies

$$\frac{\rho \circ \Theta}{\rho} = 1 + O(\rho^k).$$

It is a simple calculation to check that $\bar{g}(\Theta(x))_{kl} \partial_i \Theta^k \partial_j \Theta^l - \delta_{ij} = O(\rho^k)$. The lemma follows by multiplying these estimates. \square

The following proposition is a variant of [25, Lemma 5.2], see also [31, Lemma 5.1]. It states that any germ of metrics of order k can be made transverse by a unique diffeomorphism asymptotic to the identity of order $k + 1$.

Proposition 2.7. *Given an element $g \in G_k$ there exists a unique $\Theta \in I_0^{k+1}$ such that $\Theta_* g$ is transverse. Further, we have $\Theta_* g \in G_k$.*

The diffeomorphism Θ provided by this proposition is called the *adjustment diffeomorphism* for the metric g .

Proof. We are going to construct new coordinates at infinity which satisfy the transversality condition for the metric g . These new coordinates,

which we denote by (x'_1, \dots, x'_n) , will provide the required diffeomorphism Θ through the relation

$$\Theta(x_1, \dots, x_n) = (x'_1, \dots, x'_n).$$

We first set

$$\rho' = \frac{1 - |x'|^2}{2}.$$

Since we want ρ' to be close to ρ in a sense to be specified later, we set $\rho' = \rho e^v$ for a function v to be determined. We define $\tilde{g} := \rho'^2 g = e^{2v} \bar{g}$. We first impose an analog of Condition (9) upon \tilde{g} and ρ' , that is

$$|d\rho'|_{\tilde{g}}^2 = 1 - 2\rho'.$$

We rewrite this as an equation for v ,

$$\begin{aligned} 1 - 2\rho e^v &= e^{-2v} |\rho e^v dv + e^v d\rho|_{\tilde{g}}^2 \\ &= |\rho dv + d\rho|_{\tilde{g}}^2 \\ &= \rho^2 |dv|^2 + 2\rho \bar{g}(dv, d\rho) + |d\rho|_{\tilde{g}}^2, \end{aligned}$$

or

$$2\bar{g}(dv, d\rho) + \rho |dv|^2 = \frac{1 - 2\rho - |d\rho|_{\tilde{g}}^2}{\rho} + 2 - 2e^v. \quad (11)$$

Since $|\bar{g} - \delta|_{\delta} = O(\rho^k)$, we have

$$|d\rho|_{\tilde{g}}^2 = 1 - 2\rho + O(\rho^k),$$

meaning that

$$\frac{1 - 2\rho - |d\rho|_{\tilde{g}}^2}{\rho} = O(\rho^{k-1}). \quad (12)$$

Equation (11) is a first order partial differential equation for v . The relevant theory for such equations can be found in [16, Chapter 2] or in [33, Theorem 22.39]. In particular, the condition $|d\rho|_{\tilde{g}}^2 \equiv 1$ on \mathbb{S}^{n-1} ensures that there exists a unique solution v in a neighborhood of \mathbb{S}^{n-1} such that $v = 0$ on \mathbb{S}^{n-1} .

From Estimate (12) it follows that $v = O(\rho^k)$. We have now determined the function $\rho' = e^v \rho$.

We introduce the analog of Equation (8) for the coordinates x'^i ,

$$\tilde{g}(d\rho', dx'^i) = -x'^i,$$

with the boundary condition $x'^i = x^i$ on \mathbb{S}^{n-1} . As for the previous equation, existence and uniqueness of a smooth solution in a neighborhood of \mathbb{S}^{n-1} is guaranteed by classical results. From a simple calculation it follows that

$$\tilde{g}(d\rho', dx^i) = -x^i + O(\rho'^k).$$

This implies that $x'^i - x^i = O(\rho'^{k+1})$ which means that $\Theta \in I_0^{k+1}$. It remains to check that $\rho' = \frac{1-|x'|^2}{2}$. Setting $\rho'' = \frac{1-|x'|^2}{2}$ we have

$$\begin{aligned}\tilde{g}(d\rho', d\rho'') &= -x'^i \tilde{g}(d\rho', dx'^i) \\ &= \sum_i (x'^i)^2 \\ &= 1 - 2\rho''.\end{aligned}$$

This equation is a non-characteristic first order partial differential equation for the function ρ'' . Since x'^i coincides with x^i on \mathbb{S}^{n-1} we have that $\rho'' = 0 = \rho'$ on \mathbb{S}^{n-1} . It follows that $\rho'' = \rho'$ in a neighborhood of \mathbb{S}^{n-1} .

Uniqueness is easy to prove. Assume that we have two such diffeomorphisms Θ and Θ' . Then, considering $\Psi = \Theta' \circ \Theta^{-1}$, we have to prove that for a transverse metric g the only element $\Psi \in I_0^{k+1}$ such that Ψ_*g is also transverse is the identity. Assume that the diffeomorphism Ψ is given in coordinates by

$$\Psi(x^1, \dots, x^n) = (x'^1, \dots, x'^n),$$

where $x'^i - x^i = o(\rho)$. As before we set

$$\rho = \frac{1-|x|^2}{2} \quad \text{and} \quad \rho' = \frac{1-|x'|^2}{2}.$$

From the assumption on the coordinates, we have that $\rho' = \rho + o(\rho)$. Computing as for Equation (11), we get that $v := \log \frac{\rho'}{\rho}$ vanishes on \mathbb{S}^{n-1} and satisfies

$$2\langle dv, d\rho \rangle_{\bar{g}} + \rho |dv|^2 = 2 - 2e^v,$$

where we have used the fact that $|d\rho|_{\bar{g}}^2 = 1 - 2\rho$ since g is transverse. The solution to this equation being unique, we must have $v \equiv 0$ or equivalently $\rho \equiv \rho'$. From Condition (8) we deduce that the coordinates x^i and x'^i both satisfy the transport equation

$$\langle d\rho, dx^i \rangle_{\bar{g}} = -x^i.$$

Since they coincide on \mathbb{S}^{n-1} they coincide in a neighborhood of \mathbb{S}^{n-1} .

The proof that $\Psi_*g \in G_k$ is contained in Lemma 2.6 above. \square

Remarks 2.8. • Note here that in all previous proofs of the existence of geodesic normal coordinates (see for example [31, Lemma 5.1]) there is a loss of regularity of one derivative. Avoiding this is one of the reasons why we chose to restrict our study to smooth conformally compact manifolds. However, the proof of Proposition 2.14 indicates that there should be no such loss.

- The dependence of the adjustment diffeomorphism Θ on the germ g will be studied in Subsection 2.4. The first non-trivial term in the asymptotic expansion of Θ will be computed in Proposition 2.16.

Before stating the main result of this section, we prove two important lemmas. The first lemma is a rephrasing of [13, Theorem 6.1]. It states that the germ of an asymptotic isometry can be extended as a diffeomorphism to the sphere at infinity. The proof we give is based on the theory developed in [4–6, 22]. In order to keep this section reasonably short, we refer the reader to [5] for the definition of essential sets and the relevant results concerning them.

Lemma 2.9. *For k a positive integer let $g \in G_k$ be an asymptotically hyperbolic metric and let $\Psi \in I^k(g)$ be an asymptotic isometry. If $\psi : U \rightarrow V$ represents the germ Ψ , then ψ extends to a smooth diffeomorphism $\bar{\psi} : \tilde{U} \rightarrow \tilde{V}$.*

Proof. From Proposition 2.7 it follows that we can pull back the metrics g and Ψ^*g by elements in I_0^{k+1} so that they satisfy the transversality condition. Since this corresponds to composing the diffeomorphism ψ on the left and on the right with elements of I_0^{k+1} which are smooth up to the boundary, we can assume without loss of generality that g and Ψ^*g are transverse.

From [21, Lemma 2.5.11] we know that the set $K = \{\rho \geq \varepsilon\}$ is an essential set for both g and ψ^*g provided $\varepsilon > 0$ is small enough.¹ Equivalently, $K_1 := K$ and $K_2 := \psi(K)$ are essential sets for g . Further, since the metrics g and ψ^*g are C^∞ -conformally compact, their sectional curvatures satisfy

$$\sec^g = -1 + O(\rho) \quad \text{and} \quad \sec^{\psi^*g} = -1 + O(\rho). \quad (13)$$

With some more effort one checks that

$$|\nabla^g \mathcal{R}^g|_g = -1 + O(\rho) \quad \text{and} \quad \left| \nabla^{\psi^*g} \mathcal{R}^{\psi^*g} \right|_{\psi^*g} = -1 + O(\rho). \quad (14)$$

The transversality condition imposes that the distance from K_1 (resp. K_2) with respect to the background hyperbolic metric and g (resp. ψ^*g) agree. For points y in the boundary of $\{\rho \geq \varepsilon\}$, that is such that $\rho(y) = \varepsilon$, we have $|y| = \sqrt{1 - 2\varepsilon}$. We also remark that the closest point projections $\pi(x)$ of a point x onto $\Sigma = \rho^{-1}(\{\varepsilon\})$ with respect to the metrics b , g and ψ^*g all coincide with the Euclidean closest point projection onto Σ because Σ is a round sphere centered at the origin. Hence,

$$|x - \pi(x)|^2 = (|x| - \sqrt{1 - 2\varepsilon})^2.$$

From Equation (3), the distance from K to any point x lying outside K is given by

$$\cosh d(x, K) = 1 + \frac{(|x| - \sqrt{1 - 2\varepsilon})^2}{2\varepsilon\rho(x)}.$$

¹The notion of an essential set is a priori defined only for complete manifolds. Here we can simply fill the region of $B_1(0)$ where g (resp. Ψ^*g) is not defined by an arbitrary Riemannian metric. The argument of [21, Lemma 2.5.11] depends only on the metric outside some compact set.

A straightforward calculation shows that $e^{-d(x,K)}$ can be expressed as an analytic function of ρ such that

$$e^{-d(x,K)} \sim \frac{\varepsilon}{(1 - \sqrt{1 - 2\varepsilon})^2} \rho(x). \quad (15)$$

Thus we can replace the conformal factor $e^{-d(x,K)}$ in [5] by ρ and the results of this article still apply. It follows from Equations (13), (14), and (15) that both g and Ψ^*g fulfill the conditions of [5, Theorem A] with $a = 1$. In particular, since the $C^{1,\alpha}$ -structure ($0 < \alpha < 1$) of the manifold with boundary obtained by geodesic conformal compactification is unique, we conclude that ψ extends to a $C^{1,\alpha}$ diffeomorphism $\bar{\psi} : \tilde{U} \rightarrow \tilde{V}$.

The end of the proof then follows from a standard trick involving the Christoffel symbols. Let $\bar{\Gamma}$ denote the Christoffel symbols of the metric $\bar{g} = \rho^2 g$ in the coordinate system (x^1, \dots, x^n) . Similarly, let $\tilde{\Gamma}$ denote the Christoffel symbols of the metric $\tilde{g} := \rho^2 \psi^* g$. By assumption, all the components of $\bar{\Gamma}$ and $\tilde{\Gamma}$ are smooth functions. The transformation law for the Christoffel symbols reads

$$\frac{\partial^2 \psi^k}{\partial x^i \partial x^j} = \frac{\partial \psi^l}{\partial x^i} \frac{\partial \psi^m}{\partial x^j} \bar{\Gamma}_{lm}^k - \tilde{\Gamma}_{ij}^l \frac{\partial \psi^k}{\partial x^l}.$$

Since ψ is a C^1 diffeomorphism up to the boundary, the previous formula immediately shows that ψ is actually a C^2 diffeomorphism since the right hand side only involves first order derivatives of ψ (hence C^0 functions) together with C^∞ functions (the Christoffel symbols). The process can be iterated to conclude that ψ is actually smooth up to the boundary. \square

The second lemma states that any asymptotic isometry can be written as a composition of a true isometry of the hyperbolic metric and a diffeomorphism asymptotic to the identity. This is a variant of a result by Chruściel and Nagy [14], and Chruściel and Herzlich [13]. A simplified proof (of a weaker result) can be found in [28].

Lemma 2.10. *Given a germ of metrics $g \in G_k$, any element $\Psi \in I^k(g)$ decomposes uniquely as $\Psi = \bar{A} \circ \Psi_0$ (resp. $\Psi = \Psi'_0 \circ \bar{A}$), where $A \in O_\uparrow(n, 1)$ and $\Psi_0 \in I_0^{k+1}$ (resp. $\Psi'_0 \in I_0^{k+1}$). Conversely, any element of the form $\bar{A} \circ \Psi_0$ (resp. $\Psi'_0 \circ \bar{A}$), where $\Psi_0 \in I_0^{k+1}$ (resp. $\Psi'_0 \in I_0^{k+1}$) and $A \in O_\uparrow(n, 1)$, belongs to $I^k(g)$. In particular, the set $I^k(g)$ does not depend on the choice of $g \in G_k$.*

Proof. We first prove that if A is an isometry of the metric b then A belongs to $I^k(g)$ for any metric $g \in G_k$. Any such element can be written as the composition of an element of $O(n)$ and a Lorentz boost $\bar{A}_i^s = p A_i^s p^{-1}$. Hence, from the formulas given in Subsection 2.1, we immediately see that A is actually a smooth diffeomorphism of $\bar{B}_1(0)$. The metric $\rho^2 \bar{A}_* g$ can be

rewritten as

$$\tilde{g} = \frac{\rho^2}{(\rho \circ \bar{A}^{-1})^2} \bar{A}_* \bar{g}.$$

Arguing as in the proof of Lemma 2.6, we have that the function $\frac{\rho^2}{(\rho \circ \bar{A}^{-1})^2}$ is smooth. The metric $\bar{A}_* \bar{g}$ being obviously smooth we have that \tilde{g} is a smooth metric on some \tilde{U} , $U \in \mathcal{N}_\infty$. The condition (7) for the metric $\bar{A}_* g$ is readily checked since $r(\bar{A}(x)) = d^b(\bar{A}(x), 0) = d^b(x, \bar{A}^{-1}(0))$ (see the remark following Equation (7)).

Thus, for any $g \in G_k$, given $A \in O_\uparrow(n, 1)$ and $\Psi_0 \in I_0^{k+1}$ we have $(\Psi_0)_* g \in G_k$ so from the previous analysis $(\bar{A} \circ \Psi_0)_* g = \bar{A}_* (\Psi_0)_* g \in G_k$. A similar argument shows that $(\Psi_0 \circ \bar{A})_* g \in G_k$.

Next, given $\Psi \in I^k(g)$, we will find the element \bar{A} . It follows from Lemma 2.9 that Ψ extends to a smooth diffeomorphism up to the boundary \mathbb{S}^{n-1} . We claim that Ψ induces a conformal diffeomorphism on \mathbb{S}^{n-1} . The metric $\tilde{g} := \rho^2 \Psi_* g$ can be rewritten as

$$\tilde{g} = \frac{\rho^2}{(\rho \circ \Psi^{-1})^2} \Psi_* \bar{g}.$$

Arguing once again as in the proof of Lemma 2.6, we have that the function $\frac{\rho^2}{(\rho \circ \Psi^{-1})^2}$ is smooth on some \tilde{U} where $U \in \mathcal{N}_\infty$. Restricting to \mathbb{S}^{n-1} , since both \bar{g} and \tilde{g} restrict to the round metric σ , we have that Ψ induces a conformal isometry of \mathbb{S}^{n-1} . It follows from Liouville's theorem (see [7, Chapter A.3]) that the restriction of Ψ to \mathbb{S}^{n-1} coincide with the restriction of a unique isometry \bar{A} of the ball model of hyperbolic space.

Considering $\bar{A}^{-1} \circ \Psi$ (resp. $\Psi \circ \bar{A}^{-1}$) we are left with proving that an element $\Psi \in I^k(g)$ such that Ψ induces the identity on \mathbb{S}^{n-1} belongs to I_0^{k+1} . Composing Ψ on the left and on the right by elements of I_0^{k+1} we can assume that both metrics g and $\Psi_* g$ are transverse. The lemma will follow if we can prove that, under these assumptions, Ψ is the identity.

Following the proof of Proposition 2.7, it suffices to show that the ratio $\frac{\rho \circ \Psi}{\rho}$ tends to 1 at infinity. We set $\Xi := \Psi^{-1}$ and write

$$\Xi^j(x) = x^j + \rho \xi^j(x)$$

together with $\bar{g} = \rho^2 g = \delta + \bar{e}$ where $|\bar{e}|_\delta = O(\rho^k)$. Note that

$$\partial_i \Xi^k = \delta_i^k + (\partial_i \rho) \xi^k + \rho \partial_i \xi^k = \delta_i^k - x^i \xi^k + O(\rho).$$

The condition $\Psi_*g = \Xi^*g \in G_k$ can be written in coordinates as

$$\begin{aligned}\rho^{-2}(\Xi(x))(\delta_{kl} + \bar{e}_{kl}(\Xi(x)))\partial_i\xi^k\partial_j\xi^l &= \rho^{-2}\delta_{ij} + O(\rho^{k-2}), \\ \delta_{kl}\left(\delta_i^k - x^i\xi^k\right)\left(\delta_j^l - x^j\xi^l\right) &= \frac{\rho^2(\Xi(x))}{\rho^2}\delta_{ij} + O(\rho) \\ \delta_{ij} - x^i\xi^j - x^j\xi^i + x^ix^j|\xi|^2 &= \frac{\rho^2(\Xi(x))}{\rho^2}\delta_{ij} + O(\rho).\end{aligned}$$

The limit of $\frac{\rho \circ \Psi}{\rho}$ on \mathbb{S}^{n-1} is then obtained by considering the last equality contracted twice with any vector V orthogonal to the position vector field P , that is $V^ix^i = 0$,

$$\delta_{ij}V^iV^j = \frac{\rho^2(\Xi(x))}{\rho^2}\delta_{ij}V^iV^j + O(\rho).$$

Hence,

$$\left.\frac{\rho \circ \Psi}{\rho}\right|_{\mathbb{S}^{n-1}} = 1.$$

This concludes the proof of the lemma. \square

One of the consequences of Lemma 2.10 is that the set $I^k(g)$ is independent of $g \in G_k$ since any of its elements can be written as $\bar{A} \circ \Psi_0$ where \bar{A} is an isometry of b and $\Psi_0 \in I_0^{k+1}$. It can also be seen that $I^k(g)$ is actually a group under composition. Indeed, since $I^k(g)$ is independent of $g \in G_k$, given $\Psi_1, \Psi_2 \in I^k(g)$, we have $(\Psi_2)_*g \in G_k$ and

$$(\Psi_1 \circ \Psi_2)_*g = (\Psi_1)_*((\Psi_2)_*g) \in G_k,$$

thus proving that $I^k(g)$ is closed under composition.

The relationship between the groups of isometries, asymptotic isometries, and diffeomorphisms asymptotic to the identity is summarized in the following theorem.

Theorem 2.11. *There is a short exact sequence of groups*

$$0 \longrightarrow I_0^{k+1} \xrightarrow{i} I^k(g) \xrightarrow{\pi} O_{\uparrow}(n, 1) \longrightarrow 0. \quad (16)$$

Proof. It follows from Lemma 2.6 that we have an inclusion i of I_0^{k+1} in $I^k(g)$. Given any element $\Theta \in I_0^{k+1}$ and any $\Psi \in I^k(g)$, Lemma 2.10 gives us a decomposition $\Psi = \bar{A} \circ \Psi_0$ where $\Psi_0 \in I_0^{k+1}$ and $A \in O_{\uparrow}(n, 1)$. Then

$$\Psi \circ \Theta \circ \Psi^{-1} = \bar{A} \circ \Psi_0 \circ \Theta \circ \Psi_0^{-1} \circ \bar{A}^{-1}.$$

Note that $\Psi_0 \circ \Theta \circ \Psi_0^{-1} \in I_0^{k+1}$ so there exists an element $\Theta_1 \in I_0^{k+1}$ such that $\bar{A} \circ \Psi_0 \circ \Theta \circ \Psi_0^{-1} = \Theta_1 \circ \bar{A}$. Consequently,

$$\Psi \circ \Theta \circ \Psi^{-1} = \Theta_1 \circ \bar{A} \circ \bar{A}^{-1} = \Theta_1 \in I_0^{k+1}.$$

This proves that I_0^{k+1} is a normal subgroup of $I^k(g)$, and the quotient $I^k(g)/I_0^{k+1}$ is therefore identified with $O_{\uparrow}(n, 1)$. \square

The sequence (16) actually splits which can be seen by taking $g = b$ (and recalling that $I^k(g)$ does not depend on g). The natural action of $O_{\uparrow}(n, 1)$ on \mathbb{H}^n gives an embedding of $O_{\uparrow}(n, 1)$ into $I^k(b)$ which is a right inverse for π .

Theorem 2.11 allows us to construct a natural action of $O_{\uparrow}(n, 1)$ on the set G_k^T of transversal germs of metrics.

Proposition 2.12. *There exists a unique action of $O_{\uparrow}(n, 1)$ on the set G_k^T such that, for any $A \in O_{\uparrow}(n, 1)$ and $g \in G_k^T$, we have*

$$A \cdot g = \tilde{A}_* g,$$

where \tilde{A} is the unique element in $I^k(g)$ such that $\tilde{A}_* g$ is transverse and $\pi(\tilde{A}) = A$.

Proof. Let $g \in G_k^T$ and $A \in O_{\uparrow}(n, 1)$. From Proposition 2.7, there exists a unique $\Theta \in I_0^{k+1}$ such that the metric $\Theta_* \bar{A}_* g$ is transverse. We set $A \cdot g := \Theta_* \bar{A}_* g$. Note that $A_g := \Theta \circ \bar{A}$ is the unique element of $I^k(g)$ such that $\pi(A_g) = A$ and $(A_g)_* g \in G_k^T$.

We check that this defines an action of $O_{\uparrow}(n, 1)$ on G_k^T . The property $\text{Id} \cdot g = g$ is immediate. Let $A, B \in O_{\uparrow}(n, 1)$ and $g \in G_k^T$ be given. Note that $\pi(A_{B \cdot g} B_g) = \pi(A_{B \cdot g}) \pi(B_g) = AB$ and

$$(A_{B \cdot g})_*(B_g)_* g = (A_{B \cdot g})_*(B \cdot g) \in G_k^T.$$

From the discussion above, we conclude that $A_{B \cdot g} \circ B_g = (AB)_g$ and hence

$$A \cdot (B \cdot g) = (A_{B \cdot g})_*(B_g)_* = ((AB)_g)_* g = (AB) \cdot g.$$

□

2.3. Jets of asymptotically hyperbolic metrics. The linear masses we define should be functions of the germ at infinity of an asymptotically hyperbolic metric. There is however an important caveat preventing us from using such a definition. Namely, we want to impose some continuity assumption for the invariants, and the problem is then that the stalk of asymptotically hyperbolic metrics has no natural topology. For this reason we introduce the set of l -jets of asymptotically hyperbolic metrics.

Definition 2.13. For positive integers k, l with $l > k$, we define the set of l -jets of asymptotically hyperbolic metrics of order k as

$$J_k^l := G_k / \sim_{l+1}$$

where the equivalence relation \sim_{l+1} is defined by $g_1 \sim_{l+1} g_2$ if and only if $|g_1 - g_2|_b = O(\rho^{l+1})$. We denote the projection from G_k to J_k^l by Π_k^l . A jet $j \in J_k^l$ is called *transverse* if there exists a germ $g \in G_k$ representing j which is transverse. We denote the set of transverse jets in J_k^l by T_k^l .

The topology on the set of l -jets is defined as follows. Passing to polar coordinates, an asymptotically hyperbolic metric g defined in an open subset of the form $B_1(0) \setminus \overline{B}_{1-\varepsilon}(0)$ can be viewed as a 1-parameter curve $(0, \varepsilon) \ni \rho \mapsto g(\rho)$ where $g(\rho)$ is a smooth metric on the bundle $T\mathbb{R}^n|_{\mathbb{S}^{n-1}}$, that is a smooth positive definite section of the bundle $\text{Sym}^2(T^*\mathbb{R}^n|_{\mathbb{S}^{n-1}})$, with the further property that the map $\rho \mapsto \rho^2 g(\rho)$ extends smoothly to $\rho = 0$. The Levi-Civita connection of \mathbb{R}^n induces a connection on $\text{Sym}^2(T^*\mathbb{R}^n|_{\mathbb{S}^{n-1}})$ which allows us to define the standard Fréchet space topology on the space of smooth sections $\Gamma(\text{Sym}^2(T^*\mathbb{R}^n|_{\mathbb{S}^{n-1}}))$. The relevant theory of Fréchet spaces can be found in [26, 40].

In this terminology, an asymptotically hyperbolic germ can be thought as a germ at $\rho = 0$ of curves $\rho \mapsto g(\rho)$ defined on an interval of the form $(0, \varepsilon)$ such that $\bar{g}(\rho) := \rho^2 g(\rho)$ extends smoothly to the interval $[0, \varepsilon)$. In polar coordinates the hyperbolic metric reads

$$b = \rho^{-2} \left(\frac{d\rho^2}{1-2\rho} + (1-2\rho)\sigma \right).$$

A metric g is then transverse if $g - b$ is a 1-parameter family of sections of $\text{Sym}^2(T^*\mathbb{S}^{n-1})$ extended trivially in the ρ -direction. The set of l -jets of asymptotically hyperbolic metrics is thus identified with $l + 1$ copies of $\Gamma(\text{Sym}^2(T^*\mathbb{R}^n|_{\mathbb{S}^{n-1}}))$ via

$$g \mapsto \left(\bar{g}(0), \partial_\rho \bar{g}(0), \dots, \partial_\rho^l \bar{g}(0) \right).$$

We define the topology on J_k^l as the product topology through this identification.

We now relate jets of asymptotically hyperbolic metrics to the theory developed in Subsection 2.2.

Proposition 2.14. *Given $g \in G_k$, we let Θ be the adjustment diffeomorphism corresponding to g (see Proposition 2.7). For any $l > k$, the $(l + 1)$ -jet of Θ is fully determined by the l -jet of g and depends smoothly on it.*

Thus the Taylor expansion of the diffeomorphism Θ can be obtained formally from the Taylor expansion of the metric $\bar{g} = \rho^2 g$. From this proposition it follows that if the l -jet of the metric g is transverse, then the $(l + 1)$ -jet of the adjustment diffeomorphism Θ is trivial. We will use this fact Remark 2.17.

Proof of Proposition 2.14. Set $\Psi = \Theta^{-1}$. We will show that the $(l + 1)$ -jet of Ψ is determined by the l -jet of g . We assume that $\Psi \in I_0^{k+1}$ is such that $\Psi^* g$ is transverse,

$$\rho^2 \Psi^*(\rho^{-2} \bar{g})_{ij} x^i = x^j.$$

Introducing “polar” coordinates (ρ, φ^A) , where (φ^A) are coordinates on an open subset of the sphere \mathbb{S}^{n-1} , the transversality condition reads

$$\rho^2 \Psi^*(\rho^{-2} \bar{g})(\partial_\rho, \cdot) = \rho^2 b(\partial_\rho, \cdot) = \frac{d\rho}{1-2\rho}.$$

We use the index 0 to denote the ρ direction and capital letter indices ranging from 1 to $n-1$ to denote directions tangent to the sphere. Using this convention, the transversality condition can be rephrased as

$$\bar{g}_{ab}(\Psi(x))\partial_0\Psi^a(x)\partial_j\Psi^b(x) = \left(\frac{\rho(\Psi(x))}{\rho(x)}\right)^2 \frac{\delta_j^0}{1-2\rho}.$$

Since we assumed that $|\Psi(x) - x| = O(\rho^{k+1})$, with $k \geq 1$, we have

$$\Psi^0(x) = \rho(x) + \psi^0(x), \quad \Psi^A(x) = \varphi^A(x) + \psi^A(x),$$

with $\psi^0(x), \psi^A(x) = O(\rho^{k+1})$. Introducing this into the transversality condition, we get

$$\left(1 + \frac{\psi^0}{\rho}\right)^2 \frac{\delta_j^0}{1-2\rho} = \bar{b}_{ab}\partial_0\Psi^a(x)\partial_j\Psi^b(x) + (\bar{g}_{ab}(\Psi(x)) - \bar{b}_{ab})\partial_0\Psi^a(x)\partial_j\Psi^b(x)$$

We rewrite this equation in the case $j = 0$ and in the case $j = C \neq 0$,

$$\begin{aligned} \left(1 + \frac{\psi^0}{\rho}\right)^2 \frac{1}{1-2\rho} &= \frac{1}{1-2\rho} (1 + \partial_0\psi^0)^2 + (1-2\rho)\sigma_{AB}\partial_0\psi^A\partial_0\psi^B \\ &\quad + (\bar{g}_{ab}(\Psi(x)) - \bar{b}_{ab})\partial_0\Psi^a(x)\partial_0\Psi^b(x), \end{aligned} \quad (17a)$$

$$\begin{aligned} 0 &= \frac{1}{1-2\rho} (1 + \partial_0\psi^0) \partial_C\psi^0 + (1-2\rho)\sigma_{AB}\partial_0\psi^A(\delta_C^B + \partial_C\psi^B) \\ &\quad + (\bar{g}_{ab}(\Psi(x)) - \bar{b}_{ab})\partial_0\Psi^a(x)\partial_C\Psi^b(x). \end{aligned} \quad (17b)$$

The first equation can be rewritten as

$$\begin{aligned} &\left(2 + \frac{\psi^0}{\rho} + \partial_0\psi^0\right) \left(\frac{\psi^0}{\rho} - \partial_0\psi^0\right) \\ &= (1-2\rho)^2\sigma_{AB}\partial_0\psi^A\partial_0\psi^B \\ &\quad + (1-2\rho)(\bar{g}_{ab}(\Psi(x)) - \bar{b}_{ab})\partial_0\Psi^a(x)\partial_0\Psi^b(x). \end{aligned} \quad (17a')$$

We next make a Taylor expansion in ρ of the unknowns ψ^0, ψ^A ,

$$\begin{cases} \psi^0 = \psi_2^0\rho^2 + \dots + \psi_{l+1}^0\rho^{l+1} + O(\rho^{l+2}), \\ \psi^A = \psi_2^A\rho^2 + \dots + \psi_{l+1}^A\rho^{l+1} + O(\rho^{l+2}). \end{cases} \quad (18)$$

Note that the first two terms in the Taylor expansion disappear since we assumed that $\psi^i = O(\rho^2)$. The proof now goes by induction on l . The point is that, having determined ψ^0 and ψ^A up to order ρ^l , we can determine the coefficient ψ_{l+1}^0 by expanding Equation (17a') up to order ρ^l . The only place where ψ_{l+1}^0 shows up in (17a') is in

$$\frac{\psi^0}{\rho} - \partial_0\psi^0 = -\rho\psi_2^0 - 2\rho^2\psi_3^0 - \dots - l\rho^l\psi_{l+1}^0 + O(\rho^{l+1}).$$

We can then determine ψ_{l+1}^A by looking at Equation (17b). Also here, ψ_{l+1}^A shows up only at one place, namely in $(1-2\rho)\sigma_{AB}\partial_0\psi^A\delta_A^B$.

It is important at each step to notice that in order to determine the coefficients ψ_{l+1}^0 and ψ_{l+1}^A , we only need the Taylor expansion of the coefficients \bar{g}_{ab} up to order ρ^l , and that the coefficients of the Taylor expansions (18) are actually polynomials in \bar{g}_{ab} and its derivatives at $\rho = 0$. The details of the proof being pretty messy, we leave them to the interested reader. \square

Proposition 2.14 also implies that the “transversalization” operation via the adjustment diffeomorphism descends to jets. This is the content of the next proposition.

Proposition 2.15. *There exists a map*

$$\theta : J_k^l \rightarrow T_k^l$$

with the property that the diagram

$$\begin{array}{ccc} G_k & \xrightarrow{\Theta_*} & G_k^T \\ \downarrow \Pi_k^l & & \downarrow \Pi_k^l \\ J_k^l & \xrightarrow{\theta} & T_k^l \end{array},$$

commutes. Here Θ_ is (with some abuse of notation) the map associating to a given element $g \in G_k$ the element $\Theta_*g \in G_k^T$, where $\Theta \in I_0^{k+1}$ is the adjustment diffeomorphism defined in Proposition 2.7. Assuming that $l < 2k$, the map θ is affine.*

Proof. The proof of the first part of the proposition follows directly from Proposition 2.14. Left to prove is that θ is affine.

Let g_0 and g_1 be two elements of G_k . For $\lambda \in [0, 1]$, we set

$$g_\lambda := (1 - \lambda)g_0 + \lambda g_1$$

and denote by Ψ_λ the unique element in I_0^{k+1} for which $\Psi_\lambda^* g_\lambda$ is transverse. We wish to show that

$$\Psi_\lambda^* g_\lambda = (1 - \lambda)\Psi_0^* g_0 + \lambda\Psi_1^* g_1,$$

at least at the level of l -jets. Since the right hand side is linear in λ it is enough to show that

$$\rho^2 \frac{d^2}{d\lambda^2} \Psi_\lambda^* g_\lambda = 0, \tag{19}$$

once again at the level of l -jets. We are going to perform formal calculations, not worrying about the fact that Ψ_λ depends in a C^2 -manner on λ but from the previous proposition, we know that the l -jet of Ψ_λ depends smoothly on λ .

We will compute this second order derivative at some $\lambda_0 \in [0, 1]$. To simplify calculations, we can replace the metrics g_0 and g_1 by $\Psi_{\lambda_0}^* g_0$ and $\Psi_{\lambda_0}^* g_1$ and hence assume that Ψ_{λ_0} is the identity and that g_{λ_0} is transverse.

Using component notation we compute at $\lambda = \lambda_0$,

$$\begin{aligned}
\rho^2 \frac{d^2}{d\lambda^2} (\Psi_\lambda^* g_\lambda)_{ij} &= 6 \frac{\partial_a \rho}{\rho} \frac{\partial_b \rho}{\rho} \Psi_\lambda'^a \Psi_\lambda'^b \bar{g}_{\lambda ij}(x) - 2 \frac{\partial_a \partial_b \rho}{\rho} \Psi_\lambda'^a \Psi_\lambda'^b \bar{g}_{\lambda ij}(x) - 2 \frac{\partial_a \rho}{\rho} \Psi_\lambda''^a \bar{g}_{\lambda ij}(x) \\
&\quad - 4 \frac{\partial_a \rho}{\rho} \Psi_\lambda'^a \bar{g}'_{\lambda ij} - 4 \frac{\partial_a \rho}{\rho} \Psi_\lambda'^a \partial_b \bar{g}_{\lambda ij} \Psi_\lambda'^b \\
&\quad - 4 \frac{\partial_a \rho}{\rho} \Psi_\lambda'^a \bar{g}_{\lambda kj}(x) \partial_i \Psi_\lambda'^k - 4 \frac{\partial_a \rho}{\rho} \Psi_\lambda'^a \bar{g}_{\lambda il}(x) \partial_j \Psi_\lambda'^l \\
&\quad + 2 \partial_a \bar{g}'_{\lambda ij} \Psi_\lambda'^a + 2 \bar{g}'_{\lambda kj} \partial_i \Psi_\lambda'^k + 2 \bar{g}'_{\lambda il} \partial_j \Psi_\lambda'^l \\
&\quad + \partial_a \partial_b \bar{g}_{\lambda ij} \Psi_\lambda'^a \Psi_\lambda'^b + \partial_a \bar{g}_{\lambda ij} \Psi_\lambda''^a \\
&\quad + 2 \partial_a \bar{g}_{\lambda kj} \Psi_\lambda'^a \partial_i \Psi_\lambda'^k + 2 \partial_a \bar{g}_{\lambda il} \Psi_\lambda'^a \partial_j \Psi_\lambda'^l \\
&\quad + \bar{g}_{\lambda kj}(x) \partial_i \Psi_\lambda''^k + \bar{g}_{\lambda il}(x) \partial_j \Psi_\lambda''^l + 2 \bar{g}_{\lambda kl}(x) \partial_i \Psi_\lambda'^k \partial_j \Psi_\lambda'^l.
\end{aligned}$$

Since $\Psi', \Psi'' \in I_0^{k+1}$, it follows by inspection of the decay order of each term that

$$\rho^2 \frac{d^2}{d\lambda^2} (\Psi_\lambda^* g_\lambda)_{ij} = -2 \frac{\partial_a \rho}{\rho} \Psi_\lambda''^a \bar{g}_{\lambda ij}(x) + \bar{g}_{\lambda kj}(x) \partial_i \Psi_\lambda''^k + \bar{g}_{\lambda il}(x) \partial_j \Psi_\lambda''^l + O(\rho^{2k}).$$

To prove that the remaining three terms are also $O(\rho^{2k})$, we need to take a closer look at Equations (17a') and (17b),

$$\begin{aligned}
2 \left(\frac{\psi^0}{\rho} - \partial_0 \psi^0 \right) &= (1 - 2\rho)(\bar{g}_{00}(\Psi(x)) - \bar{b}_{00}) + O(\rho^{2k}) \\
&= (1 - 2\rho)(\bar{g}_{00}(x) - \bar{b}_{00} + \partial_a \bar{b}_{00} \psi^a(x) + \partial_a (\bar{g}_{00} - \bar{b}_{00}) \psi^a(x)) + O(\rho^{2k}), \\
&= (1 - 2\rho)(\bar{g}_{00}(x) - \bar{b}_{00} + \partial_a \bar{b}_{00} \psi^a(x)) + O(\rho^{2k}), \quad (17a'') \\
0 &= \frac{1}{1 - 2\rho} \partial_C \psi^0 + (1 - 2\rho) \sigma_{AC} \partial_0 \psi^A + (\bar{g}_{0C}(\Psi(x)) - \bar{b}_{0C}) + O(\rho^{2k}), \\
&= \frac{1}{1 - 2\rho} \partial_C \psi^0 + (1 - 2\rho) \sigma_{AC} \partial_0 \psi^A + \bar{g}_{0C}(x) + O(\rho^{2k}), \quad (17b'')
\end{aligned}$$

where we used the fact that $\bar{b}_{0C} = 0$ in the second calculation. It follows by induction that ψ depends linearly on $\bar{g} - \bar{b}$ up to terms of order $O(\rho^{2k})$. This implies that $\Psi_\lambda'' = O(\rho^{2k})$, so

$$\Pi_k^l \left(\frac{d^2}{d\lambda^2} \Psi_\lambda^* g_\lambda \right) = 0$$

provided $l < 2k$. □

In the next proposition we compute the first non-trivial term in the asymptotic expansion of $\Theta_* g$.

Proposition 2.16. *Let $g \in G_k$ be any given metric. We denote by m the first non-trivial term in the asymptotic expansion of g , that is*

$$g = b + \rho^{k-2}m + O(\rho^{k-1})$$

where m is a section of $\text{Sym}^2(T^*\mathbb{R}^n|_{\mathbb{S}^{n-1}})$. Denoting by Θ the adjustment diffeomorphism of g from Proposition 2.7, the metric $\tilde{g} := \Theta_*g$ has the asymptotic expansion

$$\tilde{g} = b + \rho^{k-2}\tilde{m} + O(\rho^{k-1}),$$

where the section \tilde{m} of $\text{Sym}^2(T^*\mathbb{S}^{n-1})$ is given by

$$\tilde{m}_{ij} = m_{ij} - m_{aj}x^ax_i - m_{ia}x^ax_j + m_{ab}\frac{x^ax^b}{k}((k-1)x_ix_j + \delta_{ij}). \quad (20)$$

Proof. Calculations starting from Equations (17a) and (17b) are fairly easy. Indeed, the first non-trivial term in the asymptotic expansion of $\Psi = \Theta^{-1}$ is the one of order ρ^{k+1} . They can be obtained looking at the terms of order $O(\rho^k)$ in (17a)-(17b),

$$\psi_{k+1}^0 = -\frac{1}{2k}m_{00}, \quad \sigma_{AB}\psi^B = -\frac{m_{0B}}{k+1}.$$

Hence, from the identity

$$\rho^2\tilde{g} = \left(\frac{\rho}{\rho \circ \Psi}\right)^2 \Psi^*\bar{g}$$

we find that

$$\tilde{m}_{00} = 0, \quad \tilde{m}_{0A} = 0, \quad \tilde{m}_{AB} = m_{AB} + \frac{m_{00}}{k}\sigma_{AB}.$$

The relation between m and \tilde{m} can be condensed into

$$\tilde{m} = m - d\rho \otimes m(\cdot, \partial_\rho) - m(\partial_\rho, \cdot) \otimes d\rho + \frac{m(\partial_\rho, \partial_\rho)}{k}(\delta + (k-1)d\rho \otimes d\rho).$$

We now wish to return to Cartesian coordinates. To do this, it suffices to note that they are related to the (ρ, φ^A) -coordinates via

$$x^i = \sqrt{1 - 2\rho F^i}(\varphi),$$

where (F^i) is a set of n given functions such that $\sum (F^i)^2 = 1$, meaning that $F^i = \frac{x^i}{|x|}$. As a consequence,

$$\frac{\partial}{\partial \rho} = \frac{\partial x^i}{\partial \rho} \frac{\partial}{\partial x^i} = -\frac{1}{\sqrt{1-2\rho}} F^i \frac{\partial}{\partial x^i} = -\frac{1}{|x|^2} x^i \frac{\partial}{\partial x^i}.$$

Formula (20) follows. \square

Remark 2.17. Returning to the proof of Proposition 2.15, it is important to note at this point that we can gain in the order up to which θ is affine by restricting ourselves to metrics that are already transverse up to some high order. Indeed, from Proposition 2.14, if $g \in J_k^l \cap T_k^{l'}$, we have that the

adjustment diffeomorphism Θ is in $I_0^{l'+2}$. Hence, following the lines of the proof of Proposition 2.15, we have that

$$\theta : J_k^l \cap T_k^{l'} \rightarrow T_k^l$$

is affine as long as $l \leq k + l'$.

The following theorem will allow us to define asymptotic invariants.

Theorem 2.18. *There exists a unique action of the group $O_\uparrow(n, 1)$ on T_k^l such that the projection $\Pi_k^l : G_k^T \rightarrow T_k^l$ is a $O_\uparrow(n, 1)$ -equivariant map. Namely,*

$$A \cdot \left(\Pi_k^l(g) \right) = \Pi_k^l(A \cdot g),$$

for all $A \in O_\uparrow(n, 1)$ and all $g \in G_k^T$, where the action of $O_\uparrow(n, 1)$ on G_k^T was defined in Proposition 2.12. Further the action is linear and smooth as long as $l \leq 2k$ and reduces in the case $l = k$ to the pushforward action

$$A \cdot g = \overline{A}_* g,$$

where $A \in O_\uparrow(n, 1)$ is any hyperbolic isometry.

Proof. The only non-trivial point in the proof is the fact that the action is linear for all $l \leq 2k$. This is where Remark 2.17 turns out to be important. We shall see that for $g \in T_k^l$ and $A \in O_\uparrow(n, 1)$ we have $\overline{A}_* g \in T_k^k$, meaning that transversality of the first non-trivial term in the asymptotic expansion of g is preserved under the (non-adjusted) action of $O_\uparrow(n, 1)$. From Remark 2.17 it then follows that $g \mapsto \theta \overline{A}_* g$ is affine for $l \leq 2k$.

As before we denote by $P = x^i \partial_i$ the position vector field and we set $g = b + e$. Since A is an hyperbolic isometry, we have

$$\overline{A}_* g = b + \overline{A}_* e.$$

Transversality of g reads $e(P, \cdot) = 0$. So we need to check that

$$\left| \rho^2 (\overline{A}_* e)(P, \cdot) \right|_\delta = \left| \rho^2 \overline{A}_* \left(e(\overline{A}^* P, \cdot) \right) \right|_\delta = O(\rho^{k+1}).$$

This will follow by showing that $\overline{A}^* P = \lambda(x, A)P + O(\rho)$ for some function $\lambda(x, A)$. Before entering calculations, we note that this fact is natural from the point of view of hyperbolic geometry. Indeed, the action of an hyperbolic isometry can be understood as a change of origin of the hyperbolic space and P is a vector field pointing in the direction of geodesics emanating from the origin. All hyperbolic geodesics intersect the boundary \mathbb{S}^{n-1} orthogonally. Hence, a tensor that is transverse with respect to some choice of an origin remains transverse up to correction terms with respect to the new origin.

The condition

$$\overline{A}^* P = \lambda(x, A)P + O(\rho) \tag{21}$$

only needs to be checked for generators of the Lorentz group. It is obvious for rotations since $\overline{R}^* P = P$ for any rotation R . The case of Lorentz boosts requires some more calculations.

With A replaced by A^{-1} , Condition (21) is equivalent to

$$\overline{A}_*P = \lambda(x, A^{-1})P + O(\rho),$$

or to

$$d\overline{A}(P)(x) = \lambda(\overline{A}(x), A^{-1})P(\overline{A}(x)) + O(\rho).$$

We use Formula (5) to write

$$\begin{aligned} d\overline{A}_i^s(x) &= \frac{1}{D} (dx^1, \dots, \cosh(s)dx^i + \sinh(s)x^j dx^j, \dots, dx^n) \\ &\quad - \frac{1}{D^2} \left(x^1, \dots, \cosh(s)x^i + \sinh(s)\frac{1+|x|^2}{2}, \dots, x^n \right) \\ &\quad \cdot ((\cosh(s) - 1)x^j dx^j + \sinh(s)dx^i), \end{aligned}$$

so

$$\begin{aligned} d\overline{A}^s(P) &= \left(\frac{1}{D} - \frac{(\cosh(s) - 1)|x|^2 + \sinh(s)x^i}{D^2} \right) \\ &\quad \cdot \left(x^1, \dots, \cosh(s) + \sinh(s)\frac{1+|x|^2}{2}, \dots, x^n \right) + O(\rho) \\ &= \frac{1}{\cosh(s) + \sinh(s)x^i} P(\overline{A}_i^s(x)) + O(\rho), \end{aligned}$$

and the claim follows. \square

2.4. Linear masses at infinity. We now come to the main definition of this article.

Definition 2.19. Let V be a finite dimensional representation of the group $O_{\uparrow}(n, 1)$. A *linear mass at infinity* for the set of asymptotically hyperbolic metrics of order k is a map $\Phi : G_k \rightarrow V$ such that:

1. $\Phi(b) = 0$.
2. For any $g \in G_k$ and any $\Psi \in I^k(g)$ it holds that

$$\Phi(\Psi \cdot g) = \pi(\Psi) \cdot \Phi(g).$$

That is, Φ is an intertwining map.

3. There exists $l \geq k$ such that Φ factors through T_k^l ,

$$\begin{array}{ccc} G_k & \xrightarrow{\Phi} & V \\ & \searrow \tilde{\Pi}_k^l & \nearrow \varphi \\ & T_k^l & \end{array}$$

where $\tilde{\Pi}_k^l$ denotes the composition $\Pi_k^l \circ \Theta_*$, and Θ_* is (with abuse of notation) the map which takes an element $g \in G_k$ to the unique transverse element $\Theta_*g \in G_k^T$ where $\Theta \in I_0^{k+1}$.

4. The affine map φ is continuous.

Let us make a few comments on this definition.

- Remarks 2.20.*
- At this point we could be more general assuming only that the space V is a $O_\uparrow(n, 1)$ -set. The restriction to vector space representations is not relevant to some large extent since we can embed any reasonable $O_\uparrow(n, 1)$ -set into a (linear) representation of $O_\uparrow(n, 1)$, see [39, Chapter 7, Section 1.3]. The only non-trivial point is that the map $\Phi : G_k \rightarrow V$ has to be linear. Invariants depending polynomially on the leading term in the asymptotic expansion of $g - b$ (the so-called mass aspect) have been recently introduced, see e.g. [34].
 - We called these objects masses instead of “invariants” since we strictly speaking do not get something independent of the chosen chart at infinity. The situation is the same for the classical mass. If one insists on having a true invariant, one has to look at the $O_\uparrow(n, 1)$ -orbit to which $\Phi(g)$ belongs. This relies on the classical invariant theory described in [39, Chapter 11] or [23, Chapter 5] and will be addressed in a forthcoming paper.
 - The condition $\Phi(b) = 0$ is imposed since we want our invariants to measure the difference between some given metric and the hyperbolic one. This condition will be immediately fulfilled if we assume that V has no 1-dimensional trivial subrepresentation, in particular if V is an irreducible representation with $\dim V > 1$. Indeed, the hyperbolic metric is a fixed point for the action of $O_\uparrow(n, 1)$ and under the condition stated above, the only such fixed point in V has to be 0.
 - It will follow from Proposition 5.1 that it actually suffices to look for such maps Φ which factorize through T_k^k . This is the approach we use in Section 3.

We will now turn to the strategy for the classification of these geometric masses.

3. ACTION OF HYPERBOLIC ISOMETRIES ON MASS-ASPECT TENSORS

We are going to classify all linear masses at infinity as defined in Definition 2.19. Our way of finding such maps $\Phi : G_k \rightarrow V$ is to start by looking for continuous maps

$$\varphi : T_k^k \rightarrow V,$$

which are intertwining for the action of the group $O_\uparrow(n, 1)$, where V is a finite-dimensional irreducible representation of the group $O_\uparrow(n, 1)$.

Note that whenever such a map φ is found, one immediately gets from Theorem 2.18 that the map $\Phi : G_k \rightarrow V$ defined as

$$\Phi = \varphi \circ \tilde{\Pi}_k^k$$

satisfies the requirements of Definition 2.19. The result of Proposition 5.1 finally makes sure that we find all possible linear masses at infinity this way.

On the way to describe the action of the orthochronous Lorentz group $O_\uparrow(n, 1)$ on such (k -jets of) metrics, we notice that it is in fact equivalent to describe the action on the *mass-aspect tensors*, that is the first non-trivial

term \tilde{m} in the asymptotic expansion of the metric \tilde{g} in Proposition 2.16, see Lemma 3.2.

One motivation for considering directly the group action (and intertwining maps) on the set of mass aspect tensors is that it allows one to read the linear masses at infinity as expressions computed directly at the conformal boundary of the asymptotically hyperbolic manifold (here the sphere with its standard conformal class), and not through a limit process. This is precisely the idea of the definition of the mass vector by Wang in [43].

Let us denote by $S^2(\mathbb{S}^{n-1}) := \Gamma(\text{Sym}^2(T^*\mathbb{S}^{n-1}))$ the set of symmetric $(2,0)$ -tensors on \mathbb{S}^{n-1} , also called *mass-aspect tensors*. After equipping it with an $O_\uparrow(n,1)$ -action, it follows from Proposition 3.3 that our quest for linear masses at infinity reduces to looking for $O_\uparrow(n,1)$ -intertwining maps

$$\Phi : S^2(\mathbb{S}^{n-1}) \longrightarrow V$$

sending a mass-aspect tensor to an element of a finite-dimensional representation V of $O_\uparrow(n,1)$.

For the sake of simplicity, we shall first find such maps Φ which are intertwining with respect to the Lorentz Lie algebra $\mathfrak{so}(n,1)$. Note that any $O_\uparrow(n,1)$ -representation V is naturally a $\mathfrak{so}(n,1)$ -representation as well.

The set of $\mathfrak{so}(n,1)$ -intertwining maps from $S^2(\mathbb{S}^{n-1})$ to V is larger than the set of such maps which are intertwining for the action of the group $O_\uparrow(n,1)$. We will however argue that all the maps we find in Section 4 are genuine $O_\uparrow(n,1)$ -intertwining, hence give linear masses at infinity.

3.1. Action of $O_\uparrow(n,1)$ on mass-aspect tensors. Let g be a k -jet of metrics in T_k^k . As before, we will abuse notation and also write g for a representative metric in this class. Such metric g is then identified with its principal part of the asymptotic expansion,

$$g = b + \rho^{k-2}m \mod T_k^k,$$

where $m \in S^2(\mathbb{S}^{n-1})$. We say that m is the *mass-aspect tensor* of g . We now identify T_k^k and $S^2(\mathbb{S}^{n-1})$ as follows.

Lemma 3.1. *For each $k \geq 1$, the map $S^2(\mathbb{S}^{n-1}) \longrightarrow T_k^k$ which maps m to $g = b + \rho^{k-2}m$ is bijective.*

Proof. Remember that the space of l -jets J_k^l is identified with the product of $l+1$ copies of the space $\Gamma(\text{Sym}^2(T^*\mathbb{R}^n|_{\mathbb{S}^{n-1}}))$ through

$$g \mapsto (\bar{g}(0), \partial_\rho \bar{g}(0), \dots, \partial_\rho^l \bar{g}(0)),$$

hence the inverse map $T_k^k \longrightarrow S^2(\mathbb{S}^{n-1})$ we are looking for is

$$g \mapsto \frac{1}{k!} \partial_\rho^k \bar{g}(0).$$

□

We use this identification to define an $O_{\uparrow}(n, 1)$ -action on $S^2(\mathbb{S}^{n-1})$. The following lemma gives the dependence of the mass-aspect tensor of an asymptotically hyperbolic metric under the action of the group of isometries of the hyperbolic space.

Lemma 3.2. *The action of $O_{\uparrow}(n, 1)$ on T_k^k defines a unique action*

$$(A, m) \mapsto A \cdot m$$

on $S^2(\mathbb{S}^{n-1})$. Moreover, if m is the mass-aspect tensor of a metric g in T_k^k , then for every $A \in O_{\uparrow}(n, 1)$, there is a smooth, positive function $u[A]$ defined on \mathbb{S}^{n-1} such that

$$A \cdot m = u[A]^{k-2} \bar{A}_* m.$$

Proof. We apply Lemma 3.1 to the metric $A \cdot g$ in T_k^k , and we define $A \cdot m$ as the image of $A \cdot g$ in $S^2(\mathbb{S}^{n-1})$. One easily sees that this defines an $O_{\uparrow}(n, 1)$ -action on $S^2(\mathbb{S}^{n-1})$. To compute it, we have from Proposition 2.12 that $A \cdot g = \tilde{A}_* g$ with $\tilde{A} = \Theta \circ \bar{A}$, where Θ is the unique adjustment diffeomorphism in I_0^{k+1} of Proposition 2.7 associated to $\bar{A}_* g$. Both metrics $\tilde{A}_* g$ and $\bar{A}_* g$ then lie in the same class modulo T_k^k . Next, we write

$$\bar{A}_* g = b + \left(\rho \circ \bar{A}^{-1} \right)^{k-2} \bar{A}_* m + O \left((\rho \circ A^{-1})^{k-1} \right).$$

For $x \neq 0$ we set $\hat{x} := \frac{x}{|x|} \in \mathbb{S}^{n-1}$, and define

$$u[A](\hat{x}) := \lim_{\lambda \rightarrow 1} \frac{\rho(\bar{A}^{-1}(\lambda \hat{x}))}{\rho(\lambda \hat{x})}. \quad (22)$$

This defines $u[A]$ as a positive smooth function on \mathbb{S}^{n-1} . We obtain

$$\bar{A}_* g = b + \rho^{k-2} (u[A](\hat{x}))^{k-2} \bar{A}_* m + O \left(\rho^{k-1} \right).$$

Finally, we get the equality in T_k^k (hence we omit the remainder),

$$A \cdot g = b + \rho^{k-2} (u[A](\hat{x}))^{k-2} \bar{A}_* m.$$

Thus, we identify the expression of $A \cdot m$ as

$$A \cdot m = (u[A](\hat{x}))^{k-2} \bar{A}_* m.$$

□

Collecting this result together with Proposition 5.1, we obtain the statement that justifies to work with mass-aspect tensors.

Proposition 3.3. *All linear masses at infinity are obtained as intertwining maps*

$$\Phi : S^2(\mathbb{S}^{n-1}) \longrightarrow V$$

between the set of mass-aspect tensors of order k and a finite dimensional representation V of $O_{\uparrow}(n, 1)$.

Other consequences of Lemma 3.2 are the expressions of the corresponding action on the *mass-aspect function*, $\text{tr}^\sigma m$, as well as on the product $\text{tr}^\sigma m d\mu^\sigma$ of the mass-aspect function and the volume form.

Corollary 3.4. *We have*

$$A \cdot (\text{tr}^\sigma m) = (u[A])^k \text{tr}^\sigma m \circ \overline{A}^{-1},$$

and

$$A \cdot (\text{tr}^\sigma m d\mu^\sigma) = (u[A])^{k+1-n} \overline{A}_* (\text{tr}^\sigma m d\mu^\sigma).$$

Proof. The fact that $\overline{A}_* b = b$ translates into

$$(\rho \circ \overline{A}^{-1})^{-2} \overline{A}_* \delta = \rho^{-2} \delta,$$

which we can rewrite as

$$\overline{A}_* \sigma = \overline{A}_* \delta|_{|x|=1} = \lim_{|x| \rightarrow 1} \left(\frac{\rho \circ \overline{A}^{-1}(x)}{\rho(x)} \right)^2 \sigma.$$

This tells us that

$$\overline{A}_* \sigma = u[A]^2 \sigma.$$

To get the expression for the action on $\text{tr}^\sigma m$ we write

$$A \cdot g = \rho^{-2} \left[\sigma + \rho^k u[A]^{k-2}(\hat{x}) \overline{A}_* m + O(\rho^{k+1}) \right],$$

so the mass-aspect function of the metric $A \cdot g$ is

$$A \cdot (\text{tr}^\sigma m) = u[A]^{k-2}(\hat{x}) \text{tr}^\sigma(\overline{A}_* m).$$

Using the identity

$$\overline{A}_*(\text{tr}^\sigma m) = \text{tr}^{\overline{A}_* \sigma}(\overline{A}_* m) = u[A]^{-2} \text{tr}^\sigma(\overline{A}_* m),$$

we find that $\text{tr}^\sigma(\overline{A}_* m) = u[A]^2 \text{tr}^\sigma m \circ \overline{A}^{-1}$ and the result follows. For the action on the product $\text{tr}^\sigma m d\mu^\sigma$, we write

$$A_*(\text{tr}^\sigma m d\mu^\sigma) = (\text{tr}^\sigma m \circ A^{-1}) d\mu^{A_* \sigma} = (\text{tr}^\sigma m \circ A^{-1}) u[A]^{n-1} d\mu^\sigma,$$

and, again using the expression above for $A \cdot g$, we conclude that

$$A \cdot (\text{tr}^\sigma m d\mu^\sigma) = \text{tr}^\sigma(\overline{A}_* m) u[A]^{k-2} d\mu^\sigma$$

which can be rewritten as

$$u[A]^k \text{tr}^\sigma m \circ \overline{A}^{-1} d\mu^\sigma = u[A]^{k+1-n} \overline{A}_* (\text{tr}^\sigma m d\mu^\sigma),$$

as desired. \square

In particular for $k = n-1$ we have that $\text{tr}^\sigma m d\mu^\sigma$ is an invariant under the action of the group $\text{Conf}(\mathbb{S}^{n-1})$ of conformal diffeomorphisms of the sphere. This fact can be restated in terms of an action of the group $\text{Conf}(\mathbb{S}^{n-1})$ on bundles over \mathbb{S}^{n-1} of tensors with *conformal weight*, see for example [19] for more on this terminology.

3.2. Action of the Lorentz algebra on mass-aspect tensors. We now define the associated Lie algebra action of $\mathfrak{so}(n, 1)$ on mass aspect tensors in $S^2(\mathbb{S}^{n-1})$. It is given by

$$X \cdot m := \frac{\partial}{\partial s} (\overline{A}^s \cdot m) |_{s=0},$$

for $X \in \mathfrak{so}(n, 1)$ and $m \in S^2(\mathbb{S}^{n-1})$, where $(A^s)_{s \in \mathbb{R}}$ is the one-parameter subgroup of $O_\uparrow(n, 1)$ generated by X .

In the next proposition we compute this action for Lorentz boosts a_i and for rotations r_{ij} , whose expressions were introduced in Subsection 2.1.

Proposition 3.5. *Let \mathbf{a}_i be the Lorentz boost vector field as defined in (6) and \mathbf{r}_{ij} be the rotation vector field defined in (4). Then*

$$a_i \cdot m = -\nabla_{\mathbf{a}_i}^\sigma m + kx^i m, \quad (23)$$

$$r_{ij} \cdot m = -\nabla_{\mathbf{r}_{ij}}^\sigma m - (m(r_{ij}(\cdot), \cdot) + m(\cdot, r_{ij}(\cdot))), \quad (24)$$

where r_{ij} acts on vector fields tangent to \mathbb{S}^{n-1} by

$$r_{ij}(U) = U^i \partial_j - U^j \partial_i = U^i \mathbf{a}_j - U^j \mathbf{a}_i \mod (x^a \partial_a).$$

Proof. From the expression of the boost hyperbolic isometries A_i^s defined in Section 2.1 we compute

$$\rho \circ \overline{A}_i^{-s} = \frac{1}{\cosh s \frac{1+|x|^2}{1-|x|^2} - \sinh s \frac{2x^i}{1-|x|^2} + 1} = \frac{\rho}{\cosh s - x^i \sinh s} + O(\rho^2)$$

which gives us

$$u[A_i^s](\hat{x}) = \frac{1}{\cosh s - x^i \sinh s}.$$

Hence by Lemma 3.2 we have the expression

$$A_i^s \cdot m = \frac{1}{(\cosh s - x^i \sinh s)^{k-2}} (\overline{A}_i^s)_* m$$

for the action of A_i^s . We compute the derivative of this expression with respect to s , and get

$$a_i \cdot m = \frac{\partial}{\partial s} \left(\frac{1}{(\cosh s - x^i \sinh s)^{k-2}} (\overline{A}_i^s)_* m \right) |_{s=0} = -\mathcal{L}_{\mathbf{a}_i} m + (k-2)x^i m. \quad (25)$$

Next, we want to rewrite this using covariant derivatives instead of Lie derivatives. Since elements of $O_\uparrow(n, 1)$ preserve the sphere at infinity \mathbb{S}^{n-1} the vector field \mathbf{a}_i is tangent to this sphere. We have

$$\begin{aligned} (\mathcal{L}_{\mathbf{a}_i} m)(\mathbf{a}_a, \mathbf{a}_b) &= \mathbf{a}_i(m(\mathbf{a}_a, \mathbf{a}_b)) - m([\mathbf{a}_i, \mathbf{a}_a], \mathbf{a}_b) - m(\mathbf{a}_a, [\mathbf{a}_i, \mathbf{a}_b]) \\ &= (\nabla_{\mathbf{a}_i}^\sigma m)(\mathbf{a}_a, \mathbf{a}_b) + m(\nabla_{\mathbf{a}_a}^\sigma \mathbf{a}_i, \mathbf{a}_b) + m(\mathbf{a}_a, \nabla_{\mathbf{a}_b}^\sigma \mathbf{a}_i). \end{aligned}$$

We then write

$$\nabla_{\mathbf{a}_a}^\sigma \mathbf{a}_i = \nabla_{\mathbf{a}_a}^\delta \mathbf{a}_i + \delta(\mathbf{a}_a, \mathbf{a}_i) \nu$$

where $\nu = x/|x|$ is the unit outward pointing normal of \mathbb{S}^{n-1} in \mathbb{R}^n . The transversality property states that $\iota_\nu m = 0$. We need to compute $\nabla_{\mathbf{a}_a}^\delta \mathbf{a}_i$ at $|x| = 1$,

$$\nabla_{\mathbf{a}_a}^\delta \mathbf{a}_i = -x^i \partial_a + 2x^a x^i x^c \partial_c - \delta_a^i x^c \partial_c.$$

From the transversality property, this yields

$$m(\nabla_{\mathbf{a}_a}^\sigma \mathbf{a}_i, \mathbf{a}_b) = -x^i m_{ab} = -x^i m(\mathbf{a}_a, \mathbf{a}_b)$$

since we restrict ourselves on the sphere \mathbb{S}^{n-1} . Thus, we obtain

$$(\mathcal{L}_{\mathbf{a}_i} m)(\mathbf{a}_a, \mathbf{a}_b) = (\nabla_{\mathbf{a}_i}^\sigma m)(\mathbf{a}_a, \mathbf{a}_b) - 2x^i m(\mathbf{a}_a, \mathbf{a}_b),$$

for all a, b , or

$$\mathcal{L}_{\mathbf{a}_i} m = \nabla_{\mathbf{a}_i}^\sigma m - 2x^i m.$$

The right-hand side of (25) therefore is

$$-\nabla_{\mathbf{a}_i}^\sigma m + kx^i m.$$

which finally yields

$$a_i \cdot m = -\nabla_{\mathbf{a}_i}^\sigma m + kx^i m,$$

as desired.

We now derive the infinitesimal action for the one-parameter group of rotations R_{ij}^θ . Since $r_{ij} = -[a_i, a_j]$, we have

$$r_{ij} \cdot m = -[a_i, a_j] \cdot m.$$

A straightforward computation yields

$$\begin{aligned} [a_i, a_j] \cdot m &= [-\nabla_{\mathbf{a}_i}^\sigma + kx^i, -\nabla_{\mathbf{a}_j}^\sigma + kx^j]m \\ &= \nabla_{[\mathbf{a}_i, \mathbf{a}_j]}^\sigma m + \mathcal{R}^\sigma(\mathbf{a}_i, \mathbf{a}_j)m. \end{aligned}$$

For V a tangent vector to \mathbb{S}^{n-1} we have

$$\begin{aligned} \mathcal{R}^\sigma(\mathbf{a}_i, \mathbf{a}_j)V &= \sigma(\mathbf{a}_j, V)\mathbf{a}_i - \sigma(\mathbf{a}_i, V)\mathbf{a}_j \\ &= dx^j(V)\mathbf{a}_i - dx^i(V)\mathbf{a}_j \\ &= -r_{ij}(V), \end{aligned}$$

so

$$\mathcal{R}^\sigma(\mathbf{a}_i, \mathbf{a}_j)m(U, V) = m(r_{ij}(U), V) + m(U, r_{ij}V).$$

Thus we find

$$(r_{ij} \cdot m)(U, V) = -\left(\nabla_{\mathbf{r}_{ij}}^\sigma m(U, V) + m(r_{ij}(U), V) + m(U, r_{ij}(V))\right),$$

which concludes the proof of the proposition. \square

3.3. Lie algebra intertwining operators. We denote by V an arbitrary finite dimensional representation of the group of isometries $O_{\uparrow}(n, 1)$ of \mathbb{H}^n . In Lemma 3.2, we computed the action of the group $O_{\uparrow}(n, 1)$ on the mass aspect tensor corresponding to some jet of metric in T_k^k . The identification we made between the space of jets T_k^k and the mass-aspect tensors $S^2(\mathbb{S}^{n-1})$ in Lemma 3.1 leads us to seek continuous and intertwining maps

$$\Phi : S^2(\mathbb{S}^{n-1}) \rightarrow V,$$

that is continuous maps commuting with the respective $O_{\uparrow}(n, 1)$ -actions:

$$\forall (A, m) \in O_{\uparrow}(n, 1) \times S^2(\mathbb{S}^{n-1}), \quad \Phi(A \cdot m) = A \cdot (\Phi(m)).$$

For the associated Lie algebra representations this definition implies that

$$\Phi(a \cdot m) = a \cdot (\Phi(m)).$$

for any element $a \in \mathfrak{so}(n, 1)$ and any $m \in S^2(\mathbb{S}^{n-1})$.

We define an action of the Lie algebra $\mathfrak{so}(n, 1)$ on linear maps $S^2(\mathbb{S}^{n-1}) \rightarrow V$ by

$$(a \cdot \Phi)(m) := a \cdot (\Phi(m)) - \Phi(a \cdot m)$$

for $a \in \mathfrak{so}(n, 1)$ and $\Phi : S^2(\mathbb{S}^{n-1}) \rightarrow V$. Intertwining operators are then the fixed points for this action.

We choose a basis (v_μ) of V and write $\Phi = \sum_\mu \Phi^\mu v_\mu$. Then the components Φ^μ are linear forms on $S^2(\mathbb{S}^{n-1})$ which are continuous with respect to the standard topology on $S^2(\mathbb{S}^{n-1})$ from Definition 2.19, that is the Φ^μ are distributions.

As usual in distribution theory, we use the same notation for the distribution Φ^μ itself and for the distribution density with values in $S^2(\mathbb{S}^{n-1})$, so that we may write

$$\Phi(m) = \sum_\mu \int_{\mathbb{S}^{n-1}} \langle \Phi^\mu, m \rangle d\mu^\sigma v_\mu,$$

where the symbol $\langle \cdot, \cdot \rangle$ denotes the inner product induced by the metric σ on $S^2(\mathbb{S}^{n-1})$. The notation $\Phi := \sum_\mu \Phi^\mu v_\mu$ will apply for both the distribution and the $S^2(\mathbb{S}^{n-1})$ -valued density. We hope that it will be clear at any place in the sequel which object we are referring to through this symbol.

By dualizing the action of $\mathfrak{so}(n, 1)$ we next find conditions that Φ must satisfy to be an intertwining operator.

Proposition 3.6. *If $\Phi : S^2(\mathbb{S}^{n-1}) \rightarrow V$ is intertwining for the action of $\mathfrak{so}(n, 1)$ then*

$$\nabla_{a_i} \Phi + (k + 1 - n) x^i \Phi - \sum_\mu \Phi^\mu a_i \cdot v_\mu = 0 \quad (26)$$

and

$$\nabla_{r_{ij}} \Phi + \Phi(r_{ij}(\cdot), \cdot) + \Phi(\cdot, r_{ij}(\cdot)) - \sum_\mu \Phi^\mu r_{ij} \cdot v_\mu = 0. \quad (27)$$

Conversely, if (26) and (27) hold then the map $\Phi : S^2(\mathbb{S}^{n-1}) \rightarrow V$ is intertwining for the Lie algebra action.

From

$$\begin{aligned} 0 &= \int_{\mathbb{S}^{n-1}} \operatorname{div}^\sigma (\langle \Phi^\mu, m \rangle X) d\mu^\sigma \\ &= \int_{\mathbb{S}^{n-1}} (\langle \nabla_X^\sigma \Phi^\mu, m \rangle + \langle \Phi^\mu, \nabla_X^\sigma m \rangle + \langle \Phi^\mu, m \rangle \operatorname{div}^\sigma X) d\mu^\sigma \end{aligned}$$

it follows that

$$\int_{\mathbb{S}^{n-1}} \langle \Phi^\mu, \nabla_X^\sigma m \rangle d\mu^\sigma = - \int_{\mathbb{S}^{n-1}} \langle \nabla_X^\sigma \Phi^\mu + (\operatorname{div}^\sigma X) \Phi^\mu, m \rangle d\mu^\sigma, \quad (28)$$

for any vector field X on \mathbb{S}^{n-1} . Note that $\mathfrak{a}_i = \operatorname{grad}^\sigma x^i$ so $\operatorname{div}^\sigma \mathfrak{a}_i = \Delta^\sigma x^i = -(n-1)x^i$, while \mathfrak{r}_{ij} is a Killing vector field of $(\mathbb{S}^{n-1}, \sigma)$, and hence $\operatorname{div}^\sigma \mathfrak{r}_{ij} = 0$.

Proof. Assume Φ is an intertwining operator and $\mathfrak{a}_i \in \mathfrak{so}(n, 1)$ is a boost. Using (23) and (28) we have

$$\begin{aligned} 0 &= (a_i \cdot \Phi)(m) \\ &= \sum_\mu \int_{\mathbb{S}^{n-1}} \langle \Phi^\mu, m \rangle d\mu^\sigma a_i \cdot v_\mu - \int_{\mathbb{S}^{n-1}} \langle \Phi^\mu, a_i \cdot m \rangle d\mu^\sigma v_\mu \\ &= \sum_\mu \int_{\mathbb{S}^{n-1}} \langle \Phi^\mu, m \rangle d\mu^\sigma a_i \cdot v_\mu + \int_{\mathbb{S}^{n-1}} \langle \Phi^\mu, \nabla_{\mathfrak{a}_i}^\sigma m - kx^i m \rangle d\mu^\sigma v_\mu \\ &= \sum_\mu \int_{\mathbb{S}^{n-1}} \langle \Phi^\mu, m \rangle d\mu^\sigma a_i \cdot v_\mu - \int_{\mathbb{S}^{n-1}} \langle \nabla_{\mathfrak{a}_i}^\sigma \Phi^\mu + (\operatorname{div}^\sigma \mathfrak{a}_i + kx^i) \Phi^\mu, m \rangle d\mu^\sigma v_\mu \\ &= \sum_\mu \int_{\mathbb{S}^{n-1}} \langle \Phi^\mu, m \rangle d\mu^\sigma a_i \cdot v_\mu - \int_{\mathbb{S}^{n-1}} \langle \nabla_{\mathfrak{a}_i}^\sigma \Phi^\mu - (n-1-k)x^i \Phi^\mu, m \rangle d\mu^\sigma v_\mu \end{aligned}$$

for $i = 1, \dots, n$. Since this holds for all $m \in S^2(\mathbb{S}^{n-1})$ we find that

$$\nabla_{\mathfrak{a}_i} \Phi - (n-1-k)x^i \Phi - \sum_\mu \Phi^\mu a_i \cdot v_\mu = 0.$$

To compute the action of a rotation $\mathbf{r}_{ij} \in \mathfrak{so}(n, 1)$ we need the following formula where ε_A denotes an orthonormal frame on \mathbb{S}^{n-1} ,

$$\begin{aligned}
\langle \Phi, m(r_{ij}(\cdot), \cdot) \rangle &= \sum_{A,B} \Phi(\varepsilon_A, \varepsilon_B) m(r_{ij}(\varepsilon_A), \varepsilon_B) \\
&= \sum_{A,B,C} \Phi(\varepsilon_A, \varepsilon_B) m(\langle r_{ij}(\varepsilon_A), \varepsilon_C \rangle \varepsilon_C, \varepsilon_B) \\
&= \sum_{A,B,C} \langle \varepsilon_A, r_{ij}^*(\varepsilon_C) \rangle \Phi(\varepsilon_A, \varepsilon_B) m(\varepsilon_C, \varepsilon_B) \\
&= - \sum_{A,B,C} \Phi(\langle \varepsilon_A, r_{ij}(\varepsilon_C) \rangle \varepsilon_A, \varepsilon_B) m(\varepsilon_C, \varepsilon_B) \\
&= - \sum_{B,C} \Phi(r_{ij}(\varepsilon_C), \varepsilon_B) m(\varepsilon_C, \varepsilon_B) \\
&= - \langle \Phi(r_{ij}(\cdot), \cdot), m \rangle,
\end{aligned}$$

and similarly,

$$\langle \Phi, m(\cdot, r_{ij}(\cdot)) \rangle = - \langle \Phi(\cdot, r_{ij}(\cdot)), m \rangle.$$

Hence (24) together with (28) tells us that

$$\begin{aligned}
0 &= (r_{ij} \cdot \Phi)(m) \\
&= \sum_{\mu} \int_{\mathbb{S}^{n-1}} \langle \Phi^{\mu}, m \rangle d\mu^{\sigma} r_{ij} \cdot v_{\mu} - \int_{\mathbb{S}^{n-1}} \langle \Phi^{\mu}, r_{ij} \cdot m \rangle d\mu^{\sigma} v_{\mu} \\
&= \sum_{\mu} \int_{\mathbb{S}^{n-1}} \langle \Phi^{\mu}, m \rangle d\mu^{\sigma} r_{ij} \cdot v_{\mu} + \int_{\mathbb{S}^{n-1}} \langle \Phi^{\mu}, \nabla_{\mathbf{r}_{ij}}^{\sigma} m + m(r_{ij}(\cdot), \cdot) + m(\cdot, r_{ij}(\cdot)) \rangle d\mu^{\sigma} v_{\mu} \\
&= \sum_{\mu} \int_{\mathbb{S}^{n-1}} \langle \Phi^{\mu}, m \rangle d\mu^{\sigma} r_{ij} \cdot v_{\mu} - \int_{\mathbb{S}^{n-1}} \langle \nabla_{\mathbf{r}_{ij}}^{\sigma} \Phi^{\mu} + \Phi^{\mu}(r_{ij}(\cdot), \cdot) + \Phi^{\mu}(\cdot, r_{ij}(\cdot)), m \rangle d\mu^{\sigma} v_{\mu},
\end{aligned}$$

and we conclude that

$$\nabla_{\mathbf{r}_{ij}} \Phi + \Phi(r_{ij}(\cdot), \cdot) + \Phi(\cdot, r_{ij}(\cdot)) - \sum_{\mu} \Phi^{\mu} r_{ij} \cdot v_{\mu} = 0.$$

Since $r_{ij} = -[a_i, a_j]$ we see that (27) follows from (26). Since boosts a_i and rotations r_{ij} form a basis of $\mathfrak{so}(n, 1)$ it is sufficient that (26) and (27) hold to conclude that Φ is an intertwining map. \square

An important remark to make at this point is the following proposition.

Proposition 3.7. *The distributions Φ^{μ} are analytic functions.*

This follows since Φ is a solution to Equation (26) which has analytic coefficients together with the fact that the vector fields \mathbf{a}_i span the tangent space $T_p \mathbb{S}^{n-1}$ at each point p .

We end this section by looking at two simple examples of Lie algebra intertwining maps. Both turn out to be linear masses at infinity in the sense of Definition 2.19, and appear in the classification established in Section 4.

Example 3.8. As a first example, we take for V the trivial 1-dimensional representation of $\mathfrak{so}(n, 1)$. A basis consists of a single vector $v_0 \neq 0$, and the action is $a \cdot v_0 = 0$ for any $a \in \mathfrak{so}(n, 1)$. Set $\Phi = \Phi_0 v_0$. From Equation (26) evaluated at the point where $\mathfrak{a}_i = 0$ we get that $k = n - 1$. The rotations \mathfrak{r}_{AB} , $2 \leq A < B \leq n$, all vanish at the south pole $p_0 = (-1, 0, \dots, 0)$, so Equation (27) evaluated at this point yields

$$\Phi_0(r_{AB}(\cdot), \cdot) + \Phi_0(\cdot, r_{AB}(\cdot)) = 0.$$

This means that Φ_0 is a bilinear form which is invariant under the action of $\mathfrak{so}(n - 1)$. The only such forms are proportional to the metric σ . It follows that $\Phi = \sigma v$ for some $v \in V$ at the south pole. By rotational symmetry this extends to $\Phi = \sigma v$ on all of \mathbb{S}^{n-1} , where $v : \mathbb{S}^{n-1} \rightarrow V$ is smooth. From (26) it follows that v is constant, and $\Phi = \sigma v$ for some $v \in V$.

Conversely it is clear that $\Phi = \sigma v$ satisfies Equations (26) and (27) with $k = n - 1$. Going back to the expression of the $\mathfrak{so}(n, 1)$ -intertwining map in this case, we obtain

$$\Phi : m \longmapsto \int_{\mathbb{S}^{n-1}} \text{tr}^\sigma m \, d\mu^\sigma v.$$

The equivariance of such a map under the full group of hyperbolic isometries $O_\uparrow(n, 1)$ holds due to Corollary 3.4 for $k = n - 1$. From Proposition 3.3, we conclude that the number $\int_{\mathbb{S}^{n-1}} \text{tr}^\sigma m \, d\mu^\sigma$ is a linear mass at infinity. This extends to the non-Einstein case (with conformal infinity $(\mathbb{S}^{n-1}, [\sigma])$) the notion of *conformal anomaly*, see for example [19, Chapter 3], for general Poincaré-Einstein manifolds.

Example 3.9. As a second example, we choose $V = \mathbb{R}^{n,1}$ with the standard action of $O_\uparrow(n, 1)$. Let v_0, \dots, v_n be the standard orthonormal basis of V . For $k = n$, Wang defines in [43] the *energy-momentum vector* of g as

$$\Phi(m) = \int_{\mathbb{S}^{n-1}} \text{tr}^\sigma m \, d\mu^\sigma v_0 + \sum_i \int_{\mathbb{S}^{n-1}} x^i \text{tr}^\sigma m \, d\mu^\sigma v_i. \quad (29)$$

Here the x^i 's are the coordinate functions on \mathbb{R}^n restricted to \mathbb{S}^{n-1} . In the work of Wang as well as in the work of Chruściel-Herzlich [13, 43], it is proven that the group $O_\uparrow(n, 1)$ acts on such vectors so that Φ is actually an $O_\uparrow(n, 1)$ -intertwining map, hence a linear mass at infinity from Proposition 3.3. Further, the Minkowski norm of this vector does not depend on the choice of the asymptotically hyperbolic chart. Whenever it is non-negative, it is the square of the so-called *asymptotically hyperbolic mass*.

4. CLASSIFICATION OF LINEAR MASSES AT INFINITY

In this section we will classify all maps $\Phi : S^2(\mathbb{S}^{n-1}) \rightarrow V$ which are intertwining for the action of the Lie group $O_\uparrow(n, 1)$. These maps are in particular $\mathfrak{so}(n, 1)$ -intertwining.

We first reduce the classification to a representation theoretic problem involving only finite-dimensional representations. This is done in Subsections 4.1 and 4.2 using the following steps.

- The Lie algebra $\mathfrak{so}(n, 1)$ has a parabolic subalgebra \mathfrak{p} consisting of all elements for which the associated vector field vanishes at the south pole $p_0 = (-1, 0, \dots, 0)$. This corresponds to the parabolic subgroup P of $SO_{\uparrow}(n, 1)$ fixing the south pole of the sphere at infinity. We first study the necessary conditions on Φ coming from Equations (26) and (27) with vector fields fixing the south pole. The derivative terms in those equations vanish at the south pole, and we obtain a set of algebraic equations for $\Phi(p_0)$. These equations are stated in Proposition 4.1.
- In Subsections 4.3 and 4.4 we use methods from representation theory to classify all solutions to the equations at the south pole.
- Using the $O_{\uparrow}(n, 1)$ -intertwining property of Φ , in particular for $SO(n)$ as described in Subsection 4.2, we will be able to deduce the expression of $\Phi(x)$ at any point $x \in \mathbb{S}^{n-1}$.

The conclusions of this part of the argument is collected in Proposition 4.5. It remains to check that the $\mathfrak{so}(n, 1)$ -intertwining operators do yield linear masses at infinity according to Definition 2.19. The first step is in Theorems 4.2, 4.4, 4.5, where we show that the $\mathfrak{so}(n, 1)$ -intertwining operators we have found lift to $O_{\uparrow}(n, 1)$ -intertwining operators on the set of mass-aspect tensors. Proposition 3.3 will then ensure that we obtain the final classification of linear masses at infinity.

4.1. Elements fixing the south pole. The subalgebra \mathfrak{p} of $\mathfrak{so}(n, 1)$ fixing the south pole has a basis consisting of

- the infinitesimal boost a_1 in the x^1 -direction,
- infinitesimal translations at the north pole $s_A := a_A + r_{1A}$, $2 \leq A \leq n$,
- infinitesimal rotations r_{AB} , $2 \leq A < B \leq n$.

Let (v_{μ}) be a basis of the $\mathfrak{so}(n, 1)$ -representation V .

Proposition 4.1. *Suppose $\Phi : S^2(\mathbb{S}^{n-1}) \rightarrow V$ is an $\mathfrak{so}(n, 1)$ -intertwining map. Then $\Phi(p_0) = \sum_{\mu} \Phi^{\mu}(p_0)v_{\mu}$ satisfies*

$$\sum_{\mu} \Phi^{\mu}(p_0)(a_1 \cdot v_{\mu}) = (n - 1 - k)\Phi(p_0), \quad (30)$$

$$\sum_{\mu} \Phi^{\mu}(p_0)s_A \cdot v_{\mu} = 0, \quad (31)$$

$$\sum_{\mu} \Phi^{\mu}(p_0)r_{AB} \cdot v_{\mu} = \Phi(p_0)(r_{AB}(\cdot, \cdot)) + \Phi(p_0)(\cdot, r_{AB}(\cdot)). \quad (32)$$

Proof. Equation (26) for a_1 evaluated at the south pole $p_0 = (-1, 0, \dots, 0)$ gives us

$$\begin{aligned} 0 &= \nabla_{\mathfrak{a}_1} \Phi(p_0) - (n-1-k)x^1 \Phi(p_0) - \sum_{\mu} \Phi^{\mu}(p_0) a_1 \cdot v_{\mu} \\ &= (n-1-k) \Phi(p_0) - \sum_{\mu} \Phi^{\mu}(p_0) a_1 \cdot v_{\mu}. \end{aligned}$$

For translations at the north pole, $s_A = a_A + r_{1A}$, we get

$$\begin{aligned} 0 &= \nabla_{\mathfrak{a}_A} \Phi - (n-1-k)x^A \Phi - \sum_{\mu} \Phi^{\mu} a_A \cdot v_{\mu} \\ &\quad + \nabla_{r_{1A}} \Phi + \Phi(r_{1A}(\cdot), \cdot) + \Phi(\cdot, r_{1A}(\cdot)) - \sum_{\mu} \Phi^{\mu} r_{1A} \cdot v_{\mu} \\ &= \nabla_{s_A} \Phi - (n-1-k)x^A \Phi - \sum_{\mu} \Phi^{\mu} s_A \cdot v_{\mu} \\ &\quad + \Phi(r_{1A}(\cdot), \cdot) + \Phi(\cdot, r_{1A}(\cdot)). \end{aligned}$$

from (26) and (27). At the south pole, the vector field $\mathfrak{s}_A := \mathfrak{a}_A + \mathfrak{r}_{1A}$ vanishes and $x^A = 0$. Further, the tangent space at the south pole is spanned by ∂_C , $2 \leq C \leq n$, and

$$r_{1A}(\partial_C) = dx^1(\partial_C) \partial_A - dx^A(\partial_C) \partial_1 = -\delta_C^A \partial_1,$$

so $\Phi(r_{1A}(\cdot), \cdot) = 0$. Together we find that

$$\sum_{\mu} \Phi^{\mu}(p_0) s_A \cdot v_{\mu} = 0.$$

Finally, Equation (27) evaluated at the south pole tells us that

$$\begin{aligned} 0 &= \nabla_{\mathfrak{r}_{AB}} \Phi(p_0)(p_0) + \Phi(p_0)(r_{AB}(\cdot), \cdot) + \Phi(p_0)(\cdot, r_{AB}(\cdot)) - \sum_{\mu} \Phi^{\mu}(p_0) r_{AB} \cdot v_{\mu} \\ &= \Phi(p_0)(r_{AB}(\cdot), \cdot) + \Phi(p_0)(\cdot, r_{AB}(\cdot)) - \sum_{\mu} \Phi^{\mu}(p_0) r_{AB} \cdot v_{\mu}. \end{aligned}$$

□

4.2. Rotations of the sphere. For $x \in \mathbb{S}^{n-1}$, let B_x be the rotation in the plane spanned by x and p_0 with $B_x(p_0) = x$. Since the map B_x is a rotation, the corresponding function $u[B]$ of Lemma 3.2 is equal to 1, and it therefore acts on the space of mass-aspect tensors $S^2(\mathbb{S}^{n-1})$ by $B_x \cdot m = (B_x)_* m$. As a result,

$$\Phi(B_x \cdot m) = \sum_{\mu} \int_{\mathbb{S}^{n-1}} \langle \Phi^{\mu}, (B_x)_* m \rangle d\mu^{\sigma} v_{\mu}.$$

Since B_x is an isometry of the sphere \mathbb{S}^{n-1} , we have

$$\begin{aligned} B_x \cdot \Phi(m) &= B_x \cdot \left(\sum_{\mu} \int_{\mathbb{S}^{n-1}} \langle \Phi^{\mu}, m \rangle d\mu^{\sigma} v_{\mu} \right) \\ &= \sum_{\mu} \int_{\mathbb{S}^{n-1}} \langle (B_x)_* \Phi^{\mu}, (B_x)_* m \rangle d\mu^{\sigma} (B_x \cdot v_{\mu}). \end{aligned}$$

The $O_{\uparrow}(n, 1)$ -intertwining property for Φ tells us that

$$\sum_{\mu} \Phi^{\mu} e_{\mu} = \sum_{\mu} (B_x)_* \Phi^{\mu} (B_x \cdot v_{\mu})$$

at all points. If we in particular evaluate at the point x we get

$$\Phi(x)(U, V) = \sum_{\mu} \Phi^{\mu}(p_0)((B_x)^* U, (B_x)^* V) (B_x \cdot v_{\mu})$$

for $U, V \in T_x \mathbb{S}^{n-1}$. This last equation gives a way to transport the expression of Φ we will obtain at the south pole p_0 to any other point of \mathbb{S}^{n-1} , thanks to the $O_{\uparrow}(n, 1)$ -intertwining property of Φ .

4.3. The 3-dimensional case. Here we specialize to the case where the dimension is $n = 3$, and assume that Φ is a $O_{\uparrow}(3, 1)$ -intertwining map. To classify all possibilities for Φ we start by finding all solutions to the equations at the south pole, this is Proposition 4.1.

The approach we use is specific to the 3-dimensional case, and to a certain degree also the results are specific, compare Theorem 4.4 with the higher-dimensional version stated in Theorem 4.5. The methods used in this section are rather elementary and might serve as a warm-up for the higher-dimensional case which is treated in Subsection 4.4.

It is a standard fact that the complexified Lie algebra $\mathfrak{so}(3, 1) \otimes \mathbb{C}$ can be written as the sum of two copies of $\mathfrak{sl}(2)$. Several such splittings exist since $\mathfrak{sl}(2)$ has many automorphisms. We implement a splitting by choosing generators

$$\begin{cases} h_1 &= -a_1 - ir_{23} \\ e_1 &= \frac{1}{2}(-a_2 + ia_3 - r_{12} + ir_{13}) \\ f_1 &= \frac{1}{2}(-a_2 - ia_3 + r_{12} + ir_{13}) \end{cases}, \quad \begin{cases} h_2 &= -a_1 + ir_{23} \\ e_2 &= -a_2 - ia_3 - r_{12} - ir_{13} \\ f_2 &= \frac{1}{4}(-a_2 + ia_3 + r_{12} - ir_{13}) \end{cases}.$$

The families $\{h_1, e_1, f_1\}$ and $\{h_2, e_2, f_2\}$ each satisfy the commutation relations of $\mathfrak{sl}(2)$, and they commute with each other. Complex irreducible representations of the Lie algebra $\mathfrak{so}(3, 1)$ are therefore in bijection with tensor product representations $V^1 \otimes V^2$ where V^1 (resp. V^2) is an irreducible representation of $(\mathfrak{sl}_2)_1 = \text{vect}(h_1, e_1, f_1)$ (resp. $(\mathfrak{sl}_2)_2 = \text{vect}(h_2, e_2, f_2)$), see [23, Section 4.2.2]. Let \mathfrak{h} denote the Cartan subalgebra of $\mathfrak{so}(3, 1)$ generated by h_1 and h_2 and let $(\bar{\omega}_1, \bar{\omega}_2)$ be the basis of \mathfrak{h}^* dual to (h_1, h_2) . The highest-weight theory tells us that irreducible finite dimensional representations of $\mathfrak{so}(3, 1)$ are in bijection with the set $\mathbb{N}\bar{\omega}_1 + \mathbb{N}\bar{\omega}_2$.

Suppose that the irreducible representation V has highest weight $n_1\bar{w}_1 + n_2\bar{w}_2$. In order to characterize $\Phi(p_0)$ when $\Phi : S^2(\mathbb{S}^2) \rightarrow V$ is a $\mathfrak{so}(3,1)$ -intertwining map, we write

$$s_2 = a_2 + r_{12} = -e_1 - \frac{1}{2}e_2, \quad s_3 = a_3 + r_{13} = -i \left(e_1 - \frac{1}{2}e_2 \right).$$

From Equation (31) we find

$$e_1 \cdot (\Phi(p_0)) = e_2 \cdot (\Phi(p_0)) = 0,$$

which means that

$$\Phi(p_0) = v_{w_1\bar{w}_1 + w_2\bar{w}_2} \varphi, \quad (33)$$

where $v_{w_1\bar{w}_1 + w_2\bar{w}_2}$ is a highest weight vector of the representation $V = V_{w_1\bar{w}_1 + w_2\bar{w}_2}$ and φ is a symmetric 2-tensor over $T_{p_0}\mathbb{S}^2$. Further, Equation (30) tells us that

$$a_1 \cdot (\Phi(p_0)) = (2 - k)\Phi(p_0),$$

where a_1 is equal to $-\frac{1}{2}(h_1 + h_2)$. Therefore we have $a_1 \cdot (\Phi(p_0)) = -\frac{w_1 + w_2}{2}\Phi(p_0)$, which gives the expression

$$k = 2 + \frac{w_1 + w_2}{2}$$

for the decay rate k .

Next we use Equation (32) together with the relation $r_{23} = \frac{i}{2}(h_1 - h_2)$. At the south pole where $y = 0$ we have $r_{23}(\partial_2) = \partial_3$ and $r_{23}(\partial_3) = -\partial_2$. Evaluating (32) on all pairs of vectors tangent to the sphere \mathbb{S}^2 at the south pole, we obtain the system of equations

$$\begin{cases} \frac{i}{2}(w_1 - w_2)\varphi_{22} = 2\varphi_{23}, \\ \frac{i}{2}(w_1 - w_2)\varphi_{23} = \varphi_{33} - \varphi_{22}, \\ \frac{i}{2}(w_1 - w_2)\varphi_{33} = -2\varphi_{23}. \end{cases}$$

It is a simple exercise to solve this eigenvalue problem. Up to a multiplicative constant, we find three solutions,

- $w_1 = w_2$, with $\varphi = dx^2 \otimes dx^2 + dx^3 \otimes dx^3 = \frac{1}{2}(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$,
- $w_1 = w_2 + 4$, with $\varphi = dx^2 \otimes dx^2 - dx^3 \otimes dx^3 + i(dx^2 \otimes dx^3 + dx^3 \otimes dx^2) = dz \otimes dz$, and
- $w_1 = w_2 - 4$, with $\varphi = dx^2 \otimes dx^2 - dx^3 \otimes dx^3 - i(dx^2 \otimes dx^3 + dx^3 \otimes dx^2) = d\bar{z} \otimes d\bar{z}$,

where we have set $z := x^2 + ix^3$.

For $n_1 \geq 0$ let \mathcal{H}_{n_1} denote the space of wave harmonic homogeneous polynomials of degree n_1 on $\mathbb{R}^{3,1}$. This is a representation of $O_{\uparrow}(3,1)$ under

$$\begin{aligned} O_{\uparrow}(3,1) \times \mathcal{H}_{n_1} &\rightarrow \mathcal{H}_{n_1} \\ (A, P) &\mapsto P \circ A^{-1}. \end{aligned}$$

In the first of these three cases, we have $w_1 = w_2 = n_1 = k - 2$. We get the following family of $O_\uparrow(3, 1)$ -intertwining maps and corresponding linear masses at infinity.

Theorem 4.2 (Conformal masses). *Let $\Phi : S^2(\mathbb{S}^2) \rightarrow V$ be an $O_\uparrow(3, 1)$ -intertwining map, such that V as a $\mathfrak{so}(3, 1)$ -representation has highest weight $n_1\bar{\omega}_1 + n_1\bar{\omega}_2$, with $n_1 \geq 0$. The representation V can then be identified with the dual of \mathcal{H}_{n_1} , and the map Φ is a multiple of the map*

$$\begin{aligned} \Phi_c(m) : \mathcal{H}_{n_1} &\rightarrow \mathbb{R} \\ P &\mapsto \int_{\mathbb{S}^2} P(1, x^1, x^2, x^3) \operatorname{tr}^\sigma(m) d\mu^\sigma, \end{aligned}$$

for $m \in S^2(\mathbb{S}^2)$. Further, this map is $O_\uparrow(3, 1)$ -intertwining, hence a linear mass at infinity for asymptotically hyperbolic metrics of order $k = n_1 + 2$.

Here the linear masses from Examples 3.8 and 3.9 appear as the cases $n_1 = 0$ and $n_1 = 1$.

Proof. The representation with highest weight $n_1(\bar{\omega}_1 + \bar{\omega}_2)$ corresponding to the case $w_1 = w_2 = n_1 = k - 2$ can be realized as the subrepresentation $\operatorname{Sym}^{n_1}(\mathbb{R}^{3,1})$ of $(\mathbb{R}^{3,1})^{\otimes n_1}$ consisting of harmonic (that is, trace-free) symmetric n_1 -tensors. The highest weight vector $v_{n_1(\bar{\omega}_1 + \bar{\omega}_2)}$ is given by $(\partial_0 - \partial_1)^{\otimes n_1}$. From (33), we know that $\Phi(p_0)$ can be written up to a constant as $(\partial_0 - \partial_1)^{\otimes n_1} \sigma(p_0)$.

The vector $\partial_0 - \partial_1$ is the position vector of the south pole when we view the sphere \mathbb{S}^2 as

$$\{(1, x^1, x^2, x^3) \mid (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}. \quad (34)$$

Using the subgroup $SO(3) \subset SO_\uparrow(3, 1)$ stabilizing the vector ∂_0 to translate the formula for $\Phi(p_0)$ to any given point on \mathbb{S}^2 (see Subsection 4.2), we obtain

$$\Phi(x^1, x^2, x^3) = (\partial_0 + x^1\partial_1 + x^2\partial_2 + x^3\partial_3)^{\otimes n_1} \otimes \sigma(x^1, x^2, x^3).$$

Let $\mathbb{R}_{n_1}[X^0, X^1, X^2, X^3]$ be the space of homogeneous polynomials of degree n_1 in X^0, X^1, X^2, X^3 . Elements of the basis (∂_μ) of $\mathbb{R}^{3,1}$ act as derivations

$$\partial_\mu : \mathbb{R}_{n_1}[X^0, X^1, X^2, X^3] \rightarrow \mathbb{R}_{n_1-1}[X^0, X^1, X^2, X^3].$$

The operator $x^\mu \partial_\mu$ is a polarization operator on $\mathbb{R}_{n_1}[X^0, X^1, X^2, X^3]$, see [39, Section 2]. It follows that for any polynomial $P \in \mathbb{R}_{n_1}[X^0, X^1, X^2, X^3]$, we have

$$[\partial_0 + x^1\partial_1 + x^2\partial_2 + x^3\partial_3]^{n_1} P = (n_1)! P(1, x^1, x^2, x^3).$$

This way $\Phi(x^1, x^2, x^3)$ appears as an element of $(\mathbb{R}_{n_1}[X^0, X^1, X^2, X^3])^* \otimes \sigma$, namely

$$\Phi(m)(P) = (n_1)! \int_{\mathbb{S}^2} P(1, x^1, x^2, x^3) \operatorname{tr}^\sigma m d\mu^\sigma.$$

Notice that since $(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$ on \mathbb{S}^2 , if

$$P = ((X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2) Q,$$

we have

$$\begin{aligned} \Phi(m)(P) &= (n_1)! \int_{\mathbb{S}^2} (1 - (x^1)^2 - (x^2)^2 - (x^3)^2) Q(1, x^1, x^2, x^3) \operatorname{tr}^\sigma m d\mu^\sigma \\ &= 0. \end{aligned}$$

From the decomposition

$$\begin{aligned} \mathbb{R}_{n_1}[X^0, X^1, X^2, X^3] \\ = \mathcal{H}_{n_1} \oplus ((X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2) \mathbb{R}_{n_1-2}[X^0, X^1, X^2, X^3] \end{aligned}$$

it follows that $\Phi(m)$ is an element of $(\mathcal{H}_{n_1})^*$.

The argument to check that the maps $\Phi_c : S^2(\mathbb{S}^2) \rightarrow (\mathcal{H}_{n_1})^*$ are $O_\uparrow(3, 1)$ -intertwining is the same as in the higher-dimensional case, see the proof of Theorem 4.5. \square

For the cases $w_1 = w_2 \pm 4$, the situation is more complicated. Each representation $V_{(n_1+4)\bar{w}_1+n_1\bar{w}_2}$ and $V_{n_1\bar{w}_1+(n_1+4)\bar{w}_2}$ is a complex representation. But their sum $V_{(n_1+4)\bar{w}_1+n_1\bar{w}_2} \oplus V_{n_1\bar{w}_1+(n_1+4)\bar{w}_2}$ is a real representation. From the solution of the Clebsch-Gordan problem (see e.g. [15, Chapter X]) it follows that this combination of representations appears in the decomposition of

$$V_{n_1(\bar{w}_1+\bar{w}_2)} \otimes (V_{4\bar{w}_1} \oplus V_{4\bar{w}_2}). \quad (35)$$

The representation $V_{2\bar{w}_1} \oplus V_{2\bar{w}_2}$ can be identified with the representation $\Lambda^2 \mathbb{R}^{3,1}$ which has a basis given by $\partial_\mu \wedge \partial_\nu$, where $0 \leq \mu < \nu \leq 3$. The highest weight vectors of this representation can be computed explicitly. They are $v_{2\bar{w}_1} = (\partial_0 - \partial_1) \wedge (\partial_2 - i\partial_3)$ for $V_{2\bar{w}_1}$ and $v_{2\bar{w}_2} = (\partial_0 - \partial_1) \wedge (\partial_2 + i\partial_3)$ for $V_{2\bar{w}_2}$.

Elements in the representation $V_{4\bar{w}_1}$ (resp. $V_{4\bar{w}_2}$) can be understood as symmetric tensor products of elements in $V_{2\bar{w}_1}$ (resp. $V_{2\bar{w}_2}$). In particular, the highest weight vector of the representation $V_{4\bar{w}_1}$ (resp. $V_{4\bar{w}_2}$) is given by

$$v_{4\bar{w}_1} = ((\partial_0 - \partial_1) \wedge (\partial_2 - i\partial_3))^{\otimes 2} \quad \text{resp.} \quad v_{4\bar{w}_2} = ((\partial_0 - \partial_1) \wedge (\partial_2 + i\partial_3))^{\otimes 2}.$$

It should be noted at this point that these tensors belong to $\operatorname{Sym}^2(\Lambda^2 \mathbb{C}^4)$ and hence can be written in component notation as

$$W^{\mu\nu\alpha\beta} \partial_\mu \otimes \partial_\nu \otimes \partial_\alpha \otimes \partial_\beta.$$

It is straightforward to check that

$$\begin{aligned} 0 &= W^{\mu\nu\alpha\beta} \eta_{\mu\alpha}, \\ 0 &= W^{\mu\nu\alpha\beta} + W^{\nu\alpha\mu\beta} + W^{\alpha\mu\nu\beta}, \end{aligned}$$

which means that these vectors are trace-free with respect to the Minkowski metric η and satisfy the contravariant version of the first Bianchi identity.

Using (33) we will now look for the form of $\Phi(p_0)$. Note that, introducing the complex structure J on \mathbb{S}^2 , we have

$$\begin{aligned}(\partial_2 - i\partial_3)dz &= \partial_2 dx^2 + \partial_3 dx^3 - i(\partial_3 dx^2 - \partial_2 dx^3) = \text{Id} - iJ, \\(\partial_2 + i\partial_3)d\bar{z} &= \partial_2 dx^2 + \partial_3 dx^3 + i(\partial_3 dx^2 - \partial_2 dx^3) = \text{Id} + iJ,\end{aligned}$$

at the south pole. As a consequence, the element $v_{4\bar{\omega}_1} \otimes (dz \otimes dz)$ (resp. $v_{4\bar{\omega}_2} \otimes (d\bar{z} \otimes d\bar{z})$) can be understood as a map

$$\begin{aligned}\Psi_+ : \quad T_{p_0}\mathbb{S}^2 \otimes T_{p_0}\mathbb{S}^2 &\rightarrow \text{Sym}^2(\Lambda^2\mathbb{C}^4) \\ X \otimes Y &\mapsto ((\partial_0 - \partial_1) \wedge (X - iJ(X))) \otimes ((\partial_0 - \partial_1) \wedge (Y - iJ(Y))),\end{aligned}$$

resp.

$$\begin{aligned}\Psi_- : \quad T_{p_0}\mathbb{S}^2 \otimes T_{p_0}\mathbb{S}^2 &\rightarrow \text{Sym}^2(\Lambda^2\mathbb{C}^4) \\ X \otimes Y &\mapsto ((\partial_0 - \partial_1) \wedge (X + iJ(X))) \otimes ((\partial_0 - \partial_1) \wedge (Y + iJ(Y))),\end{aligned}$$

where $T_{p_0}\mathbb{S}^2$ denotes the tangent space at the south pole of \mathbb{S}^2 , when \mathbb{S}^2 is seen as the 2-sphere embedded in $\mathbb{R}^{3,1}$ as in (34). This formula can be written using the Hodge star operator \star acting on $\Lambda_2(\mathbb{R}^{3,1})$, see for example [8, Definition 1.51]. Straightforward computations lead to

$$\star(e_+ \wedge X) = e_+ \wedge J(X) \quad (36)$$

for any vector $X \in T_{p_0}\mathbb{S}^2$, where e_+ is the future pointing null vector at p_0 defined by $e_+ := \partial_0 - \partial_1$. Further, $\star^2 = -\text{Id}$ so the eigenvalues of \star on $\Lambda_2(\mathbb{R}^{3,1})$ are $\pm i$.

Definition 4.3. We denote by $(\Lambda_2(\mathbb{R}^{3,1}))^\pm$ the eigenspaces of the Hodge star operator on the complexified space $\Lambda_2(\mathbb{R}^{3,1}) \otimes \mathbb{C}$, that is

$$(\Lambda_2(\mathbb{R}^{3,1}))^\pm = \{\omega \in \Lambda_2(\mathbb{R}^{3,1}) \otimes \mathbb{C} \mid \star\omega = \pm i\omega\}.$$

We define p_\pm as the projection operator on $(\Lambda_2(\mathbb{R}^{3,1}))^\pm$ with respect to $(\Lambda_2(\mathbb{R}^{3,1}))^\mp$, that is $p_\pm(\omega) = \frac{1}{2}(\omega \mp i\star\omega)$.

The decomposition

$$\Lambda_2(\mathbb{R}^{3,1}) \otimes \mathbb{C} = (\Lambda_2(\mathbb{R}^{3,1}))^+ \oplus (\Lambda_2(\mathbb{R}^{3,1}))^-$$

corresponds to the decomposition into irreducible representations under the action of $\mathfrak{so}(3,1)$, where $(\Lambda_2(\mathbb{R}^{3,1}))^+$ corresponds to the representation $V_{2\bar{\omega}_1}$ and $(\Lambda_2(\mathbb{R}^{3,1}))^-$ corresponds to $V_{2\bar{\omega}_2}$.

The operators Ψ_\pm are then better understood as

$$\begin{cases} \Psi_+(X, Y) = p_+(e_+ \wedge X) \otimes p_+(e_+ \wedge Y), \\ \Psi_-(X, Y) = p_-(e_+ \wedge X) \otimes p_-(e_+ \wedge Y), \end{cases}$$

where we have dropped an irrelevant factor 4 in the definition of Ψ_\pm .

Let \mathcal{W}_0 denote the space of constant Weyl tensors, namely the set of covariant 4-tensors $W \in \text{Sym}^2(\Lambda^2\mathbb{R}^{3,1})$ satisfying

$$\eta^{\mu\alpha}W_{\mu\nu\alpha\beta} = 0, \quad W_{\mu\nu\alpha\beta} + W_{\nu\alpha\mu\beta} + W_{\alpha\mu\nu\beta} = 0.$$

Elements of \mathcal{W}_0 can be seen as endomorphisms of $\Lambda^2(\mathbb{R}^{3,1})$ commuting with \star , see [8, Paragraphs 3.19 and 3.20]. The space \mathcal{W}_0 naturally pairs with $\text{Sym}^2(\Lambda^2\mathbb{R}^{3,1})$, and since the Hodge star operator is self-adjoint we get

$$\begin{aligned}\langle W, \alpha \otimes \star \beta \rangle &= \langle \alpha, W(\star \beta) \rangle \\ &= \langle \alpha, \star W(\beta) \rangle \\ &= \langle \star \alpha, W(\beta) \rangle \\ &= \langle W, \star \alpha \otimes \beta \rangle\end{aligned}$$

for any $W \in \mathcal{W}_0$ and any $\alpha, \beta \in \Lambda_2(\mathbb{R}^{3,1})$. Pairing the elements of \mathcal{W}_0 with Ψ_\pm , we get

$$\langle W, \Psi_\pm(X, Y) \rangle = \langle W, p_\pm(e_+ \wedge X) \otimes (e_+ \wedge Y) \rangle, \quad (37)$$

where one projection p_\pm disappears due to the previous calculations.

Returning to the representations $V_{(n_1+4)\bar{\omega}_1+n_1\bar{\omega}_2}$ (resp. $V_{n_1\bar{\omega}_1+(n_1+4)\bar{\omega}_2}$), the element $v_{(n_1+4)\bar{\omega}_1+n_1\bar{\omega}_2} dz \otimes dz$, resp. $v_{n_1\bar{\omega}_1+(n_1+4)\bar{\omega}_2} d\bar{z} \otimes d\bar{z}$, can be written as

$$\Phi_\pm(p_0) = (e_+)^{\otimes n_1} \otimes (p_\pm(e_+ \wedge \cdot) \otimes p_\pm(e_+ \wedge \cdot)),$$

where the factor $(e_+)^{\otimes n_1}$ comes from the $V_{n_1(\bar{\omega}_1+\bar{\omega}_2)}$ -part of (35). Let \mathcal{W}_{n_1} denotes the set of *polynomial Weyl tensors* of degree n_1 , that is

$$\mathcal{W}_{n_1} := \{W \in \mathbb{R}_{n_1}[X^0, X^1, X^2, X^3] \otimes \mathcal{W}_0 \mid \partial_\mu W_{\nu\sigma\alpha\beta} + \partial_\nu W_{\sigma\mu\alpha\beta} + \partial_\sigma W_{\mu\nu\alpha\beta} = 0\}.$$

We extend e_+ by rotations to the null vector field

$$e_+ := x^\mu \partial_\mu = \partial_0 + x^1 \partial_1 + x^2 \partial_2 + x^3 \partial_3$$

on \mathbb{S}^2 . By invariance under rotations, and arguing as in the proof of Theorem 4.2, we have for all $W \in \mathcal{W}_{n_1}$,

$$\langle W, \Phi_\pm \rangle = (n_1)! W(e_+, (\text{Id} \mp iJ)(\cdot), e_+, \cdot)(1, x^1, x^2, x^3)$$

where the right-hand side is an element of $\mathbb{R}_{n_1}[X^0, X^1, X^2, X^3] \otimes S^2(\mathbb{S}^2)$ evaluated at the point $(X^0, X^1, X^2, X^3) = (1, x^1, x^2, x^3)$. We obtain the following Theorem.

Theorem 4.4 (Weyl masses). *Let $\Phi : S^2(\mathbb{S}^2) \rightarrow V$ be a $O_\uparrow(3, 1)$ -intertwining map, such that V , as a $\mathfrak{so}(3, 1)$ -representation has highest weight $(n_1+4)\bar{\omega}_1 + n_1\bar{\omega}_2$ (resp. $n_1\bar{\omega}_1 + (n_1+4)\bar{\omega}_2$), for $n_1 \geq 0$. The representation V can then be identified with a subspace of the dual of $\mathbb{C} \otimes \mathcal{W}_{n_1}$, and the map Φ is a multiple of the map defined by*

$$\begin{aligned}\Phi_{w,+}(m) : \mathcal{W}_{n_1} &\rightarrow \mathbb{C} \\ W &\mapsto \int_{\mathbb{S}^2} \langle m, W(e_+, (\text{Id} - iJ)(\cdot), e_+, \cdot) \rangle_\sigma d\mu^\sigma, \quad (38)\end{aligned}$$

resp.

$$\begin{aligned}\Phi_{w,-}(m) : \mathcal{W}_{n_1} &\rightarrow \mathbb{C} \\ W &\mapsto \int_{\mathbb{S}^2} \langle m, W(e_+, (\text{Id} + iJ)(\cdot), e_+, \cdot) \rangle_\sigma d\mu^\sigma \quad (39)\end{aligned}$$

for $m \in S^2(\mathbb{S}^2)$. Further, these maps are $O_\uparrow(3, 1)$ -intertwining, hence linear masses at infinity for asymptotically hyperbolic metrics of order $k = n_1 + 4$.

Note that the masses at infinity found in this theorem depend only on the trace-free part of the mass aspect tensor m .

Proof. What remains is to check that the maps $\Phi_{w,\pm}$ are $O_{\uparrow}(3,1)$ -intertwining. The argument is the same as in the higher-dimensional case, and we refer to the proof of Theorem 4.5. \square

4.4. The general case. In this section we assume $n \geq 4$. For the representation theory we follow the conventions of [23].

Recall that $(\partial_\mu)_{\mu=0,\dots,n}$ denotes the standard basis of $\mathbb{R}^{n,1}$. We introduce a basis (e_i) of $\mathbb{R}^{n,1} \otimes \mathbb{C}$ as follows.

- If $n+1 = 2l$ is even, we set $e_{+1} := -\partial_0 + \partial_1$ and $e_{-1} := \frac{\partial_0 + \partial_1}{2}$. For $k = 2, \dots, l$, we set $e_{+k} := \partial_{2k-2} + i\partial_{2k-1}$ and $e_{-k} := \frac{1}{2}(\partial_{2k-2} - i\partial_{2k-1})$.
- If $n+1 = 2l+1$ is odd, we set $e_0 := \partial_n$ and define $e_{\pm k}$, $k = 1, \dots, l$ as in the previous case.

This basis is chosen in such a way that $\eta(e_i, e_j) = \delta_{i,-j}$. We also define the dual basis (e^i) with respect to the Minkowski metric, so that $e^i = \eta(e_i, \cdot)$. We consider the maximal torus H of $SO_{\mathbb{C}}(n,1)$ which is the subgroup of matrices which are diagonal in the basis (e_i) . We denote its Lie algebra by \mathfrak{h} .

This convention is convenient since reduction from $\mathfrak{so}(n,1)$ to the subalgebra $\mathfrak{so}(n-1)$ of orientation preserving isometries at the south pole corresponds to deleting the leftmost node in the Dynkin diagrams B_l (if $n+1 = 2l+1$ is odd) or D_l (if $n+1 = 2l$ is even). See Figure 1. The number of nodes in these diagrams correspond to l , the rank of the Lie algebra $\mathfrak{so}(n,1)$.

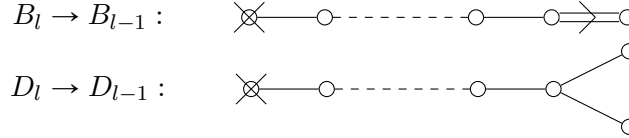


FIGURE 1. Reduction from $\mathfrak{so}(n,1)$ to $\mathfrak{so}(n-1)$ in terms of Dynkin diagrams.

Notice that for the diagram B_l with $l = 2$ and for D_l with $l = 3$ (so when $n = 4$ or 5), the shape of the Dynkin diagram changes. This means that these cases will require some special attention.

The elements s_A introduced right before Proposition 4.1 in the 3-dimensional case generalize to the ladder operator

$$\begin{cases} X_{\varepsilon_1 - \varepsilon_k} = \frac{1}{2}(s_{2k-2} + is_{2k-1}), \\ X_{\varepsilon_1 + \varepsilon_k} = s_{2k-2} - is_{2k-1}, \\ X_{\varepsilon_1} = s_n \quad (\text{for odd } n+1), \end{cases} \quad (40)$$

for $k = -l, \dots, -1, 1, \dots, l$, see [23, Section 2.4.1]. Here $\varepsilon_k \in \mathfrak{h}^*$ is the linear functional whose value on a matrix is the k -th term on its diagonal. The notation X_f where $f \in \mathfrak{h}^*$ means that $\text{ad}(h)(X_f) = f(h)X_f$, that is X_f belongs to the root space associated to f .

The fundamental weights $\bar{\omega}_i$ of $\mathfrak{so}(n, 1)$ are given by

- $(n+1 \text{ odd})$ $\bar{\omega}_i = \varepsilon_1 + \dots + \varepsilon_i$ for $1 \leq i \leq l-1$ and $\bar{\omega}_l = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_l)$,
- $(n+1 \text{ even})$ $\bar{\omega}_i = \varepsilon_1 + \dots + \varepsilon_i$ for $1 \leq i \leq l-2$ and $\bar{\omega}_{l-1} = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_{l-1} - \varepsilon_l)$, $\bar{\omega}_l = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_{l-1} + \varepsilon_l)$,

see [23, Section 3.1.3].

We let E_i be the matrix having coefficient 1 on the i -th diagonal position and zeros elsewhere. The coroots associated to the fundamental weights are given by

- $(n+1 \text{ odd})$ $H_i = E_i - E_{i+1} + E_{-i-1} - E_{-i}$ for $0 \leq i \leq l-1$ and $H_l = 2(E_l - E_{-l})$,
- $(n+1 \text{ even})$ $H_i = E_i - E_{i+1} + E_{-i-1} - E_{-i}$ for $0 \leq i \leq l-1$ and $H_l = E_{l-1} + E_l - E_{-l} - E_{-(l-1)}$.

The subalgebra $\mathfrak{so}(n-1)$ then has a Cartan subalgebra generated by H_2, \dots, H_l . Hence, the fundamental weights of $\mathfrak{so}(n-1)$ are $\bar{\omega}_i$ for $i \geq 2$ except in the case $n+1 = 5$ for which the fundamental weight is $\bar{\omega} = \varepsilon_l$ which is twice the restriction of $\bar{\omega}_2$.

We now assume that $\Phi : S^2(\mathbb{S}^{n-1}) \rightarrow V$ is an $O_\uparrow(n, 1)$ -intertwining map. Formula (32) tells us that $\Phi(p_0)$ intertwines the actions of $\mathfrak{so}(n-1)$ on $\text{Sym}^2(T_{p_0}^* \mathbb{S}^{n-1})$ and on V . Hence, the preimage of a highest weight vector v of V is a highest weight vector in $\text{Sym}^2(T_{p_0}^* \mathbb{S}^{n-1})$ for the action of $\mathfrak{so}(n-1)$.

We assume for the moment that $n \geq 6$, we will later indicate the modifications needed for $n = 4, 5$. The space $\text{Sym}^2(T_{p_0}^* \mathbb{S}^{n-1})$ decomposes into two irreducible representations for the action of $\mathfrak{so}(n-1)$,

- the trivial one-dimensional representation generated by $\sum_{A=2}^n dx^A \otimes dx^A = \sum_{j \neq \pm 1} e^j \otimes e^{-j}$,
- the set $\text{Sym}_0^2(T_{p_0}^* \mathbb{S}^{n-1})$ of symmetric trace-free 2-tensors with highest weight vector $e^2 \otimes e^2$.

The corresponding highest weights are 0 for the trivial representation and $2\bar{\omega}_2$ for $\text{Sym}_0^2(T_{p_0}^* \mathbb{S}^{n-1})$.

From the formulas in Equation (40), Condition (31) implies that if v_0 is one of the highest weight vectors given above, then $\Phi(p_0)(v_0)$ is annihilated by the full nilpotent algebra \mathfrak{n}_+ consisting of positive roots of $\mathfrak{so}(n, 1)$. In particular, up to a normalization constant, $v = \Phi(p_0)(v_0)$.

Let ω denote the highest weight of V , that is $h \cdot v = \omega(h)v$ for all $h \in \mathfrak{h}$. By classical highest weight theory, we know that $\omega = \sum_{i=1}^l n_i \bar{\omega}_i$ with non-negative integers n_i . From the discussion above, restriction to the Cartan subalgebra of $\mathfrak{so}(n-1)$ imposes that $n_2 = 0$ or 2 , $n_i = 0$ for all $i > 2$, while n_1 is arbitrary.

Condition (30) gives the decay order k of the invariant. Since $\varepsilon_1(a_1) = -1$ and $\varepsilon_2(a_1) = 0$ we have

$$\begin{aligned} (n-1-k)\Phi(p_0)(v_0) &= a_1 \cdot \left(\sum_{\mu} \Phi^{\mu}(p_0)(v_0)v_{\mu} \right) \\ &= \omega(a_1)\Phi(p_0)(v_0) \\ &= (n_1\bar{\omega}_1(a_1) + n_2\bar{\omega}_2(a_1))\Phi(p_0)(v_0) \\ &= -(n_1 + n_2)\Phi(p_0)(v_0), \end{aligned}$$

and hence $k = n - 1 + n_1 + n_2$.

The representation with highest weight $\bar{\omega}_2$ is $\Lambda^2\mathbb{R}^{n,1}$. As a consequence the representation with highest weight $\omega = n_1\bar{\omega}_1 + n_2\bar{\omega}_2$ is obtained as the submodule of highest weight $n_1\bar{\omega}_1 + n_2\bar{\omega}_2$ of

$$\underbrace{\mathbb{R}^{n,1} \otimes \dots \otimes \mathbb{R}^{n,1}}_{n_1 \text{ times}} \otimes \underbrace{\Lambda^2\mathbb{R}^{n,1} \otimes \dots \otimes \Lambda^2\mathbb{R}^{n,1}}_{n_2 \text{ times}},$$

namely the so-called Cartan product, see [23, Section 5.5.3].

It is now a good time to pause and see how the argument adapts to the cases $n = 4, 5$.

- If $n = 4$, the standard representation of $\mathfrak{so}(3)$ has highest weight $2\bar{\omega}_2$ and hence $\text{Sym}_0^2(T_{p_0}^*\mathbb{S}^{n-1})$ has highest weight $4\bar{\omega}_2$ for $\mathfrak{so}(n-1)$. When returning to $\mathfrak{so}(4,1)$ this gives again the representation $\Lambda^2(\mathbb{R}^{4,1})$ so the previous discussion applies.
- If $n = 5$, the highest weight ω corresponding to the standard representation of $SO(4)$ has $\omega(H_{l-1}) = \omega(H_l) = 1$. This is the $(1,1)$ -representation according to the previous. This lifts to the representation $\Lambda^2\mathbb{R}^{5,1}$ so the previous discussion still applies.

Since the longest element in the Weyl group acts on these highest weights by changing them to their opposite we conclude that the representations V that we found are isomorphic to their duals, see [23, Section 3.2.3].

First, we study the representation with highest weight $n_1\bar{\omega}_1$. This corresponds to the space $V = \text{Sym}_0^{n_1}(\mathbb{R}^{n,1})$ of symmetric trace-free contravariant tensors of rank n_1 . The highest weight vector of this representation is $e_{+1} \otimes \dots \otimes e_{+1}$ so the operator $\Phi(p_0)$ reads

$$\Phi(p_0) = \underbrace{e_{+1} \otimes \dots \otimes e_{+1}}_{n_1 \text{ times}} \sigma(p_0).$$

The vector e_{+1} extends by rotations to \mathbb{S}^{n-1} as the future pointing null vector $e_+ = \partial_0 + x^1\partial_1 + \dots + x^n\partial_n$. And hence, by invariance under rotations, the operator Φ reads

$$\Phi = \underbrace{e_+ \otimes \dots \otimes e_+}_{n_1 \text{ times}} \sigma.$$

As in the proof of Theorem 4.2, the space $\text{Sym}_0^{n_1}(\mathbb{R}^{n,1})$ can be identified with the dual of \mathcal{H}_{n_1} of homogeneous polynomials of degree n_1 in X^0, \dots, X^n .

Hence, the formula for these invariants takes a form similar to the one we obtained in the previous section,

$$\begin{aligned} \Phi(m) : \mathcal{H}_{n_1} &\rightarrow \mathbb{C} \\ P &\mapsto \int_{\mathbb{S}^{n-1}} P(1, x^1, \dots, x^n) \operatorname{tr}^\sigma(m) d\mu^\sigma. \end{aligned} \quad (41)$$

Second, we look at the representation with highest weight $n_1\bar{\omega}_1 + 2\bar{\omega}_2$. This can be realized as the subspace of $(\mathbb{R}^{n,1})^{\otimes(n_1+4)}$ corresponding to the Young tableau

$$Y_{n_1} = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline \end{array} \cdots \begin{array}{|c|} \hline n_1 + 4 \\ \hline \end{array}.$$

Namely, let $\mathbf{r}(Y_{n_1})$ denote the row symmetrizer

$$\mathbf{r}(Y_{n_1}) = (1 + (34)) \sum_{\sigma \in S(\{1, 2, 5, \dots, n_1+4\})} \sigma,$$

and let \mathbf{c} be the column skew symmetrizer

$$\mathbf{c}(Y_{n_1}) = (1 - (13))(1 - (24)).$$

These elements are elements of the group algebra $\mathbb{C}[S_{n_1+4}]$ and act on $(\mathbb{R}^{n,1})^{\otimes(n_1+4)}$ by permuting the positions of the elements in a simple tensor.

The Young symmetrizer $\mathbf{s}(Y_{n_1}) = \mathbf{c}(Y_{n_1})\mathbf{r}(Y_{n_1})$ is, up to a normalization constant, a projection from the subspace of harmonic tensors in $(\mathbb{R}^{n,1})^{\otimes(n_1+4)}$ onto an irreducible representation V of highest weight $n_1\bar{\omega}_1 + 2\bar{\omega}_2$. Here, as before, harmonic means that the trace with respect to any pair of indices vanishes.

It can be checked that

$$(1 + (123) + (132))\mathbf{s}(Y_{n_1}) = 0,$$

and that

$$(1 + (125) + (152))\mathbf{s}(Y_{n_1}) = 0.$$

These two identities are the (contravariant) analogs of the first and the second Bianchi identities.

The map $\Phi(p_0)$ sends the highest weight vector $e^{+2} \otimes e^{+2}$ to a highest weight vector v in V . Up to the action of an element of $\mathfrak{so}(n, 1)$, it is equal to $(e_{+1})^{\otimes n_1} \otimes (e_{+1} \wedge e_{+2}) \otimes (e_{+1} \wedge e_{+2})$, which is a highest weight vector of V . By the fact that $\Phi(p_0)$ intertwines the actions of $\mathfrak{so}(n-1)$ on $\operatorname{Sym}_0^2(T_{p_0}^* \mathbb{S}^{n-1})$ and on V , it has the form

$$\Phi(p_0)(T) = \langle T, (e_{+1})^{\otimes n_1} \otimes (e_{+1} \wedge \cdot) \otimes (e_{+1} \wedge \cdot) \rangle.$$

Note that since v is harmonic, this map extends by zero to symmetric 2-tensors proportional to $\sigma(p_0)$. The map $\Phi(p_0)$ is therefore defined by the above formula on $\operatorname{Sym}_0^2(T_{p_0}^* \mathbb{S}^{n-1})$. Due to the invariance under the action of $SO(n-1)$, the density Φ has the expression

$$\begin{aligned} \Phi : \operatorname{Sym}_0^2(T^* \mathbb{S}^{n-1}) &\rightarrow V \\ m &\mapsto \langle m, (e_+)^{\otimes n_1} \otimes (e_+ \wedge \cdot) \otimes (e_+ \wedge \cdot) \rangle. \end{aligned}$$

As in the 3-dimensional case (see Theorem 4.4), we can view Φ as acting on the space \mathcal{W}_{n_1} of polynomial Weyl tensors of degree n_1 in the variables X^0, \dots, X^n , and the corresponding intertwining maps can be written, up to a constant factor, as

$$\begin{aligned} \Phi(m) : \mathcal{W}_{n_1} &\rightarrow \mathbb{R} \\ W &\mapsto \int_{\mathbb{S}^{n-1}} \langle m, W(e_+, \cdot, e_+, \cdot) \rangle d\mu^\sigma. \end{aligned} \quad (42)$$

Here, the polynomial part of the integrand is evaluated at $(X^0, X^1, \dots, X^n) = (1, x^1, \dots, x^n)$.

We now collect the results of this section in the following theorem.

Theorem 4.5. *Let $\Phi : S^2(\mathbb{S}^{n-1}) \rightarrow V$ be an $O_\uparrow(n, 1)$ -intertwining map with $n \geq 4$, where V is an irreducible, finite dimensional representation of $O_\uparrow(n, 1)$. Then one of the following cases holds.*

- *Either V , as a $\mathfrak{so}(n, 1)$ -representation, has highest weight $n_1 \bar{\omega}_1$ with $n_1 \geq 0$. Then V can be identified with the dual of \mathcal{H}_{n_1} . The map Φ is a multiple of the map*

$$\begin{aligned} \Phi_c(m) : \mathcal{H}_{n_1} &\rightarrow \mathbb{R} \\ P &\mapsto \int_{\mathbb{S}^{n-1}} P(1, x^1, \dots, x^n) \operatorname{tr}^\sigma(m) d\mu^\sigma, \end{aligned}$$

for $m \in S^2(\mathbb{S}^{n-1})$.

- *Or V , as a $\mathfrak{so}(n, 1)$ -representation, has highest weight $n_1 \bar{\omega}_1 + 2\bar{\omega}_2$ with $n_1 \geq 0$. Then V can be identified with a subspace of the dual of \mathcal{W}_{n_1} . The map Φ is a multiple of the map*

$$\begin{aligned} \Phi_w(m) : \mathcal{W}_{n_1} &\rightarrow \mathbb{R} \\ W &\mapsto \int_{\mathbb{S}^{n-1}} \langle m, W(e_+, \cdot, e_+, \cdot) \rangle d\mu^\sigma \end{aligned}$$

Further, these maps are $O_\uparrow(n, 1)$ -intertwining, hence are linear masses at infinity, in the first case for asymptotically hyperbolic metrics of order $k = n - 1 + n_1$, in the second case, for metrics of order $k = n + 1 + n_1$.

Note that Φ_c depends only on the trace of the mass aspect tensor m , while Φ_w depends only on the trace-free part of m .

Proof. The last remaining point to prove in both cases is the $O_\uparrow(n, 1)$ -intertwining property of the maps Φ_c and Φ_w . To do this, we need to carefully keep track of the points where integration takes place. We start by a calculation for the function u defined in (22). The function $t := X^0 \circ p^{-1}$, which is the time component of a point x in the hyperbolic space, is given by

$$t(x) = \frac{1 + |x|^2}{1 - |x|^2} = \frac{1}{\rho(x)} - 1.$$

As a consequence we have

$$\begin{aligned}
u[A](\hat{x}) &= \lim_{\lambda \rightarrow 1} \frac{\rho(\bar{A}^{-1}(\lambda\hat{x}))}{\rho(\lambda\hat{x})} \\
&= \lim_{\lambda \rightarrow 1} \frac{t(\lambda\hat{x}) + 1}{t(\bar{A}^{-1}(\lambda\hat{x})) + 1} \\
&= \lim_{\lambda \rightarrow 1} \frac{t(\lambda\hat{x})}{t(\bar{A}^{-1}(\lambda\hat{x}))} \\
&= \lim_{\lambda \rightarrow 1} \left[X^0 \left(A^{-1} \cdot \frac{p^{-1}(\lambda\hat{x})}{X^0(p^{-1}(\lambda\hat{x}))} \right) \right]^{-1} \\
&= [X^0(A^{-1} \cdot (1, \hat{x}))]^{-1}
\end{aligned}$$

for any $\hat{x} \in \mathbb{S}^{n-1}$. Recall here that $\bar{A} = pAp^{-1}$ gives the action of $A \in O_{\uparrow}(n, 1)$ on the ball model of hyperbolic space. This can be rewritten as

$$A^{-1}(1, \hat{x}) = \frac{1}{u[A](\hat{x})}(1, \bar{A}^{-1}(\hat{x})).$$

From the proof of Corollary 3.4 we have

$$\bar{A}_* \sigma = u[A]^2 \sigma,$$

and thus

$$d\mu^{\bar{A}_* \sigma} = u[A]^{n-1} d\mu^{\sigma}.$$

From Lemma 3.2 we know that $A \cdot m = u[A]^{k-2} \bar{A}_* m$. So for Φ_c we get

$$\begin{aligned}
\Phi_c(A \cdot m)(P \circ A^{-1}) &= \int_{\mathbb{S}^{n-1}} P(A^{-1}(1, \hat{x})) u[A]^{k-2} \text{tr}^{\sigma}(\bar{A}_* m) d\mu^{\sigma} \\
&= \int_{\mathbb{S}^{n-1}} u[A]^{k-2-n_1} P(1, \bar{A}^{-1}\hat{x}) \text{tr}^{\sigma}(\bar{A}_* m) d\mu^{\sigma} \\
&= \int_{\mathbb{S}^{n-1}} u[A]^{k-n_1-n+1} P(1, \bar{A}^{-1}\hat{x}) \text{tr}^{\bar{A}_* \sigma}(\bar{A}_* m) d\mu^{\bar{A}_* \sigma} \\
&= \int_{\mathbb{S}^{n-1}} P(1, \bar{A}^{-1}\hat{x}) \text{tr}^{\bar{A}_* \sigma}(\bar{A}_* m) d\mu^{\bar{A}_* \sigma}
\end{aligned}$$

since P is a homogeneous polynomial of degree $k-2$, and since $k = n-1+n_1$. By a change of variables we conclude that

$$\Phi_c(A \cdot m)(P \circ A^{-1}) = \Phi_c(m)(P)$$

which is the desired $O_{\uparrow}(n, 1)$ -intertwining property of the map Φ_c .

To prove the intertwining property for Φ_w we compute

$$\begin{aligned}
\Phi_w(A \cdot m)(A_* W) &= \int_{\mathbb{S}^{n-1}} \left\langle u[A]^{k-2} \bar{A}_* m, (A_* W)(e_+, \cdot, e_+, \cdot)(1, \hat{x}) \right\rangle_{\sigma} d\mu^{\sigma}(\hat{x}) \\
&= \int_{\mathbb{S}^{n-1}} u[A]^{k-2} \left\langle \bar{A}_* m, W(A^* e_+, A^* \cdot, A^* e_+, A^* \cdot)(A^{-1}(1, \hat{x})) \right\rangle_{\sigma} d\mu^{\sigma}(\hat{x}).
\end{aligned}$$

We extend e_+ to all of $\mathbb{R}^{n,1}$ by defining it to be the position vector field, that is

$$e_+ := X^\mu \partial_\mu.$$

As a consequence we have $A^*e_+ = e_+$, and

$$\begin{aligned} \Phi_w(A \cdot m)(A_*W) \\ = \int_{\mathbb{S}^{n-1}} u[A]^{k-2} \langle \overline{A}_*m, W(e_+, A^*\cdot, e_+, A^*\cdot)(A^{-1}(1, \hat{x})) \rangle_\sigma d\mu^\sigma(\hat{x}). \end{aligned}$$

The vectors e_+ in this expression are evaluated at the point $A^{-1}(1, \hat{x})$, and we have

$$e_+(A^{-1}(1, \hat{x})) = u[A]^{-1}e_+(1, \overline{A}^{-1}\hat{x}).$$

The vectors $A^*\cdot$ are also evaluated at the point $A^{-1}(1, \hat{x})$, where \cdot is the place for a vector belonging to $T_{(1, \overline{A}^{-1}\hat{x})}\mathbb{S}^{n-1}$. Thus we get

$$(A^*\cdot)(A^{-1}(1, \hat{x})) = u[A]^{-1}(\overline{A}^*\cdot)(1, \overline{A}^{-1}\hat{x}).$$

Since W is homogeneous of degree n_1 , we find that

$$\begin{aligned} W(e_+, A^*\cdot, e_+, A^*\cdot)(A^{-1}(1, \hat{x})) \\ = u[A]^{-4-n_1}W(e_+, \overline{A}^*\cdot, e_+, \overline{A}^*\cdot)((1, \overline{A}^{-1}\hat{x})). \end{aligned}$$

We can now continue the computation,

$$\begin{aligned} \Phi_w(A \cdot m)(A_*W) \\ = \int_{\mathbb{S}^{n-1}} u[A]^{k-6-n_1} \langle \overline{A}_*m, W(e_+, \overline{A}^*\cdot, e_+, \overline{A}^*\cdot)((1, \overline{A}^{-1}\hat{x})) \rangle_\sigma d\mu^\sigma(\hat{x}) \\ = \int_{\mathbb{S}^{n-1}} u[A]^{k-n_1-n-1} \langle \overline{A}_*m, W(e_+, \overline{A}^*\cdot, e_+, \overline{A}^*\cdot)((1, \overline{A}^{-1}\hat{x})) \rangle_{\overline{A}_*\sigma} d\mu^{\overline{A}_*\sigma}(\hat{x}) \end{aligned}$$

Note that the metric σ is used twice for the scalar product, which explains the change in the exponent of $u[A]$ when passing to the last line. Since $k = n + 1 + n_1$, a change of variables gives us

$$\Phi_w(A \cdot m)(A_*W) = \Phi_w(m)(W),$$

which is the $O_\uparrow(n, 1)$ -intertwining property of the map Φ_w . \square

5. HIGHER INVARIANTS

The aim of this section is to prove the following result.

Proposition 5.1. *The only linear linear masses at infinity $\Phi : G_k \rightarrow V$ for asymptotically hyperbolic metrics of order k , in the sense of Definition 2.19, factor through T_k^k :*

$$\begin{array}{ccc} G_k & \xrightarrow{\Phi} & V \\ & \searrow \widetilde{\Pi}_k & \nearrow \varphi \\ & T_k^k & \end{array}$$

Following Definition 2.19, we assume that Φ factors through T_k^l for some $l \geq k$. Since finite dimensional representations of $O_\uparrow(n, 1)$ are completely reducible, we will also assume that V is an irreducible representation.

We notice first that there is a natural cofiltration of the set of asymptotically hyperbolic transverse germs,

$$G_1^T \supset G_2^T \supset \cdots \supset G_k^T \supset \cdots \supset G_l^T \supset G_{l+1}^T.$$

This cofiltration descends to jets,

$$T_k^l \supset T_{k+1}^l \supset \cdots \supset T_{l-1}^l \supset T_l^l.$$

The action of $O_\uparrow(n, 1)$ preserves this cofiltration. Given some metric $g \in G_k^T$, we write its asymptotic expansion as follows,

$$g = \rho^{-2} \left(\delta + g^{(k)} \rho^k + \cdots + g^{(l)} \rho^l + \cdots \right)$$

where $g^{(i)}$ is a symmetric 2-tensor on \mathbb{S}^{n-1} . By linearity, Φ can be written as

$$\Phi(g) = \varphi_k(g^{(k)}) + \cdots + \varphi_l(g^{(l)}).$$

For any $g \in G_l^T$ we have

$$\Phi(g) = \varphi_l(g^{(l)}).$$

Since Φ factors through T_k^l , its restriction to G_l^T factors through T_l^l and intertwines the action of $O_\uparrow(n, 1)$ from G_l^T to V . This means that φ_l is one of the linear masses we found in Section 4, or zero.

We now consider $\Phi - \varphi_l$. By definition this map vanishes on the set G_l^T and hence factors through T_k^{l-1} . Repeating the argument inductively, we find that each of the φ_i 's is either zero or one of the linear masses discovered in Section 4. What this means is that, using the hyperbolic metric to transform the affine spaces G_k^T into vector spaces, the linear mass Φ descends to a map from the associated graded representation to V .

The representation V is irreducible and we know from Section 4 that to each irreducible representation, we can associate at most one linear mass at a certain decay rate. This means that only a single φ_i can be non-zero. The proof of the proposition will follow from proving that $i = k$. Assume by contradiction that $i > k$. Restricting the linear mass Φ to G_{i-1}^T we can assume without loss of generality that $i = k + 1$.

We take a metric $g \in G_k^T$ which has the asymptotic expansion

$$g = \rho^{-2} \left(\delta + g^{(k)} \rho^k + g^{(k+1)} \rho^{k+1} + O(\rho^{k+2}) \right),$$

with $g^{(k+1)} \equiv 0$. The intertwining property reads:

$$\Phi(A \cdot g) = A \cdot \Phi(g) = 0.$$

The contradiction will follow if we are able to choose A and $g^{(k)}$ such that $\Phi(A \cdot g) \neq 0$. We shall however do it at the infinitesimal level for simplicity.

To this end, we explicit the action of the a_1 on T_k^{k+1} . We introduce the coordinates

$$\rho = \frac{1 - |x|^2}{2}, \quad y^A = \frac{x^A/|x|}{1 - x^1/|x|} = \frac{x^A}{|x| - x^1}.$$

Note that y^A is the stereographic projection of the radial projection of x on \mathbb{S}^{n-1} . The standard coordinates x^i can be obtained in terms of ρ and y ,

$$x^1 = \sqrt{1 - 2\rho} \frac{|y|^2 - 1}{|y|^2 + 1}, \quad x^A = \sqrt{1 - 2\rho} \frac{2y^A}{1 + |y|^2}.$$

The boost vector field $\mathfrak{a}_1 = \frac{\partial}{\partial x^1} - x^1 x^i \frac{\partial}{\partial x^i}$ can be expressed in this new coordinate system,

$$\mathfrak{a}_1 = -\rho \sqrt{1 - 2\rho} \frac{|y|^2 - 1}{|y|^2 + 1} \frac{\partial}{\partial \rho} + \frac{1 - \rho}{\sqrt{1 - 2\rho}} y^A \frac{\partial}{\partial y^A}.$$

In particular,

$$\begin{aligned} \left[\frac{\partial}{\partial \rho}, \mathfrak{a}_1 \right] &= \left(-x^1 + \frac{\rho x^1}{1 - 2\rho} \right) \frac{\partial}{\partial \rho} + \frac{\rho}{(1 - 2\rho)^{3/2}} y^A \frac{\partial}{\partial y^A}, \\ \left[\frac{\partial}{\partial y^A}, \mathfrak{a}_1 \right] &= \rho \sqrt{1 - 2\rho} \frac{4y^A}{(1 + |y|^2)^2} \frac{\partial}{\partial \rho} + \frac{1 - \rho}{\sqrt{1 - 2\rho}} \frac{\partial}{\partial y^A}. \end{aligned}$$

Rather lengthy but fairly straightforward calculations yield the following (here we do not assume that $g^{(k+1)}$ vanishes identically),

$$\begin{aligned} \mathcal{L}_{\mathfrak{a}_1} g \left(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \rho} \right) &= 0, \\ \mathcal{L}_{\mathfrak{a}_1} g \left(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial y^A} \right) &= \rho^{-2} \left(\rho^{k+1} y^B g^{(k)} \left(\frac{\partial}{\partial y^A}, \frac{\partial}{\partial y^B} \right) + O(\rho^{k+2}) \right) \\ \mathcal{L}_{\mathfrak{a}_1} g \left(\frac{\partial}{\partial y^A}, \frac{\partial}{\partial y^B} \right) &= \rho^{-2} \left(\rho^k y^C \nabla_{\frac{\partial}{\partial y^C}}^\sigma g_{AB}^{(k)} - k x^1 \rho^k g_{AB}^{(k)} \right) \\ &\quad + \rho^{-2} \left(\rho^{k+1} y^C \nabla_{\frac{\partial}{\partial y^C}}^\sigma g_{AB}^{(k+1)} - (k+1) x^1 \rho^{k+1} g_{AB}^{(k+1)} - 2\rho^{k+1} \frac{|y|^2 - 1}{|y|^2 + 1} g_{AB}^{(k)} \right) \\ &\quad + O(\rho^k). \end{aligned}$$

The last calculation being more involved, we give some details:

$$\begin{aligned}
& \mathcal{L}_{\mathbf{a}_1} g \left(\frac{\partial}{\partial y^A}, \frac{\partial}{\partial y^B} \right) \\
&= \mathcal{L}_{\mathbf{a}_1} (\rho^{-2} \bar{g}) \left(\frac{\partial}{\partial y^A}, \frac{\partial}{\partial y^B} \right) \\
&= -2 \frac{d\rho(\mathbf{a}_1)}{\rho^3} \bar{g} \left(\frac{\partial}{\partial y^A}, \frac{\partial}{\partial y^B} \right) + \rho^{-2} \mathcal{L}_{\mathbf{a}_1} \bar{g} \left(\frac{\partial}{\partial y^A}, \frac{\partial}{\partial y^B} \right) \\
&= 2x^1 g \left(\frac{\partial}{\partial y^A}, \frac{\partial}{\partial y^B} \right) + \rho^{-2} \mathbf{a}_1 \cdot \left(\bar{g} \left(\frac{\partial}{\partial y^A}, \frac{\partial}{\partial y^B} \right) \right) \\
&\quad + \rho^{-2} \bar{g} \left(\left[\frac{\partial}{\partial y^A}, \mathbf{a}_1 \right], \frac{\partial}{\partial y^B} \right) + \rho^{-2} \bar{g} \left(\frac{\partial}{\partial y^A}, \left[\frac{\partial}{\partial y^B}, \mathbf{a}_1 \right] \right) \\
&= 2x^1 \rho^{-2} \left(\frac{4(1-2\rho)\delta_{AB}}{(1+|y|^2)^2} + \rho^k g_{AB}^{(k)} + \rho^{k+1} g_{AB}^{(k+1)} + O(\rho^{k+2}) \right) \\
&\quad + \rho^{-2} \mathbf{a}_1 \cdot \left(\frac{4(1-2\rho)\delta_{AB}}{(1+|y|^2)^2} + \rho^k g_{AB}^{(k)} + \rho^{k+1} g_{AB}^{(k+1)} + O(\rho^{k+2}) \right) \\
&\quad + 2\rho^{-2} \bar{g} \left(\frac{1-\rho}{\sqrt{1-2\rho}} \frac{\partial}{\partial y^A}, \frac{\partial}{\partial y^B} \right) \\
&= 2x^1 \rho^{-2} \left(\frac{4(1-2\rho)\delta_{AB}}{(1+|y|^2)^2} + \rho^k g_{AB}^{(k)} + \rho^{k+1} g_{AB}^{(k+1)} + O(\rho^{k+2}) \right) \\
&\quad + \rho^{-2} \left(\frac{8x^1 \rho}{(1+|y|^2)^2} - \frac{16(1-\rho)\sqrt{1-2\rho}|y|^2}{(1+|y|^2)^3} \right) \delta_{AB} \\
&\quad + \rho^{-2} \left(-kx^1 \rho^k g_{AB}^{(k)} - (k+1)x^1 \rho^{k+1} g_{AB}^{(k+1)} \right. \\
&\quad \left. + \frac{1-\rho}{\sqrt{1-2\rho}} \left(\rho^k y^C \frac{\partial}{\partial y^C} g_{AB}^{(k)} + \rho^{k+1} y^C \frac{\partial}{\partial y^C} g_{AB}^{(k+1)} \right) + O(\rho^{k+2}) \right) \\
&\quad + 2\rho^{-2} \left((1-\rho)\sqrt{1-2\rho} \frac{4\delta_{AB}}{(1+|y|^2)^2} + \frac{1-\rho}{\sqrt{1-2\rho}} \left(\rho^k g_{AB}^{(k)} + \rho^{k+1} g_{AB}^{(k+1)} + O(\rho^{k+2}) \right) \right),
\end{aligned}$$

$$\begin{aligned}
& \mathcal{L}_{a_1} g \left(\frac{\partial}{\partial y^A}, \frac{\partial}{\partial y^B} \right) \\
&= 2x^1 \rho^{-2} \left(\rho^k g_{AB}^{(k)} + \rho^{k+1} g_{AB}^{(k+1)} + O(\rho^{k+2}) \right) \\
&\quad + \rho^{-2} \left(-kx^1 \rho^k g_{AB}^{(k)} - (k+1)x^1 \rho^{k+1} g_{AB}^{(k+1)} \right. \\
&\quad \left. + \frac{1-\rho}{\sqrt{1-2\rho}} \left(\rho^k y^C \frac{\partial}{\partial y^C} g_{AB}^{(k)} + \rho^{k+1} y^C \frac{\partial}{\partial y^C} g_{AB}^{(k+1)} \right) + O(\rho^{k+2}) \right) \\
&\quad + 2\rho^{-2} \left(\frac{1-\rho}{\sqrt{1-2\rho}} \left(\rho^k g_{AB}^{(k)} + \rho^{k+1} g_{AB}^{(k+1)} + O(\rho^{k+2}) \right) \right) \\
&= 2x^1 \rho^{-2} \left(\rho^k g_{AB}^{(k)} + \rho^{k+1} g_{AB}^{(k+1)} + O(\rho^{k+2}) \right) \\
&\quad + \rho^{-2} \left(-kx^1 \rho^k g_{AB}^{(k)} - (k+1)x^1 \rho^{k+1} g_{AB}^{(k+1)} \right. \\
&\quad \left. + \left(\rho^k y^C \frac{\partial}{\partial y^C} g_{AB}^{(k)} + \rho^{k+1} y^C \frac{\partial}{\partial y^C} g_{AB}^{(k+1)} \right) + O(\rho^{k+2}) \right) \\
&\quad + 2\rho^{-2} \left(\rho^k g_{AB}^{(k)} + \rho^{k+1} g_{AB}^{(k+1)} + O(\rho^{k+2}) \right),
\end{aligned}$$

$$\begin{aligned}
& \mathcal{L}_{a_1} g \left(\frac{\partial}{\partial y^A}, \frac{\partial}{\partial y^B} \right) \\
&= 2(x^1 + 1)\rho^{-2} \left(\rho^k g_{AB}^{(k)} + \rho^{k+1} g_{AB}^{(k+1)} + O(\rho^{k+2}) \right) \\
&\quad + \rho^{-2} \left(-kx^1 \rho^k g_{AB}^{(k)} - (k+1)x^1 \rho^{k+1} g_{AB}^{(k+1)} \right. \\
&\quad \left. + \left(\rho^k y^C \frac{\partial}{\partial y^C} g_{AB}^{(k)} + \rho^{k+1} y^C \frac{\partial}{\partial y^C} g_{AB}^{(k+1)} \right) + O(\rho^{k+2}) \right) \\
&= \rho^{-2} \left(\rho^k y^C \nabla_{\frac{\partial}{\partial y^C}}^\sigma g_{AB}^{(k)} + \rho^{k+1} y^C \nabla_{\frac{\partial}{\partial y^C}}^\sigma g_{AB}^{(k+1)} + O(\rho^{k+2}) \right) \\
&\quad + \rho^{-2} \left(-kx^1 \rho^k g_{AB}^{(k)} - (k+1)x^1 \rho^{k+1} g_{AB}^{(k+1)} \right) \\
&\quad + 2(\sqrt{1-2\rho} - 1) \frac{|y|^2 - 1}{|y|^2 + 1} \rho^{-2} \left(\rho^k g_{AB}^{(k)} + \rho^{k+1} g_{AB}^{(k+1)} + O(\rho^{k+2}) \right) \\
&= \rho^{-2} \left(\rho^k y^C \nabla_{\frac{\partial}{\partial y^C}}^\sigma g_{AB}^{(k)} - kx^1 \rho^k g_{AB}^{(k)} \right) \\
&\quad + \rho^{-2} \left(\rho^{k+1} y^C \nabla_{\frac{\partial}{\partial y^C}}^\sigma g_{AB}^{(k+1)} - (k+1)x^1 \rho^{k+1} g_{AB}^{(k+1)} - 2\rho^{k+1} \frac{|y|^2 - 1}{|y|^2 + 1} g_{AB}^{(k)} \right) \\
&\quad + O(\rho^k).
\end{aligned}$$

From computations similar to the proof of Proposition 3.5 we get

$$a_1 \cdot g = \rho^{-2} \left(\delta + \rho^k (a_1 \cdot g)^{(k)} + \rho^{k+1} (a_1 \cdot g)^{(k+1)} + O(\rho^{k+2}) \right),$$

where

$$\begin{aligned}(a_1 \cdot g)^{(k)} &= -y^C \nabla_{\frac{\partial}{\partial y^C}}^\sigma g_{AB}^{(k)} + kx^1 g_{AB}^{(k)} \\ (a_1 \cdot g)^{(k+1)} &= -y^C \nabla_{\frac{\partial}{\partial y^C}}^\sigma g_{AB}^{(k+1)} + (k+1)x^1 g_{AB}^{(k+1)} + 2x^1 g_{AB}^{(k)}.\end{aligned}$$

If we can choose $g^{(k)}$ so that $\varphi_l(x^1 g^{(k)}) \not\equiv 0$ we are done. The reader can pretty easily convince himself that this is indeed the case. This ends the proof of Proposition 5.1.

6. INTERPRETATION OF THE LINEAR MASSES AT INFINITY

In this section we want to give geometric interpretations of the linear masses that we have found and classified in Section 4. By this we mean geometric constructions on asymptotically hyperbolic manifolds which give the linear masses without explicitly using the asymptotic expansion of the metric and the mass aspect tensor.

The first approach is to use the framework of mass-like invariants associated to curvature type expressions developed by Michel [36], which we recall here in Subsection 6.1. It considers “geometric” differential operators

$$F : \mathcal{M}(M) \rightarrow \Gamma(E)$$

from the space $\mathcal{M}(M)$ of Riemannian metrics on a manifold M to the space $\Gamma(E)$ of sections of some tensor bundle E over M . The adjective “geometric” means that the operator is invariant under the action of diffeomorphisms, that is

$$F(\Psi_* g) = \Psi_* F(g)$$

for any diffeomorphism Ψ and $g \in \mathcal{M}(M)$.

Given a “reference space” (M_0, g_0) , for us the hyperbolic space (\mathbb{H}^n, b) , Michel showed that if the adjoint DF_b^* of the linearization of F at the hyperbolic metric b has a non-trivial kernel $\ker DF_b^* \neq \{0\}$, then there is a linear map $\Phi_F : G_F \rightarrow V_F$ which behaves like a linear mass at infinity, where $V_F = (\ker DF_b^*)^*$ is a representation of $O_\uparrow(n, 1)$ and where the space G_F consists in asymptotically hyperbolic metrics with sufficient decay rate, this rate depending on the properties of F , particularly on the growth rate of elements in $\ker DF_b^*$.

This map Φ_F is in general not a genuine linear mass at infinity, since the space $\ker DF_b^*$ may well be infinite-dimensional. Depending on the situation, we will look for finite-dimensional subspaces of $\ker DF_b^*$ which are invariant under the action of $O_\uparrow(n, 1)$.

In Subsection 6.2, we consider the example of the scalar curvature operator, precisely $F = \text{Scal} + n(n-1)$, which yields the Chruściel-Herzlich definition of the mass, from which one can recover Wang’s definition of the mass as in Example 3.9 for metrics in G_n . In Chruściel-Herzlich’s notation, the mass of a given metric g with sufficient decay is an element $H_g \in \mathbb{R}^{n,1}$ seen as a linear form on $(\mathbb{R}^{n,1})^*$, see Theorem 6.5 below.

This mass, together with the scalar curvature operator satisfy the following positive mass theorem.

Theorem 6.1 (Wang [43], Chruściel-Herzlich [13]). *Let (M, g) an asymptotically hyperbolic manifold with respect to a chart at infinity φ , such that the metric $g^\varphi := \varphi_*g$ satisfies the same assumptions as in Theorem 6.5. Assume moreover that M is spin, that the metric g is complete and has scalar curvature*

$$\text{Scal}_g \geq -n(n-1).$$

Then either the mass functional $H_g \in \mathbb{R}^{n,1}$ is timelike future-directed, or (M, g) is isometric to the hyperbolic space.

Motivated by the tight link between the scalar curvature operator F and the (standard) mass, the subsequent parts of the present section are devoted to finding, for any linear mass Φ found in Section 4, a geometric differential operator $F : \mathcal{M}(\mathbb{H}^n) \rightarrow \Gamma(E)$ for which $\Phi_F = \Phi$.

The first main motivation is to get a notion of *linear masses at infinity* which applies to more general situations than the one we discussed so far, which required that g has some asymptotic expansion of the form given in Proposition 2.16.

Another motivation for finding them is the hope to obtain a positivity theorem reminiscent of Theorem 6.1. We will however not investigate further on this topic, which we leave for a future work.

6.1. Mass invariants from geometric differential operators. In [36], Michel considers over a manifold M differential operators

$$F : \mathcal{M}(M) \rightarrow \Gamma(E)$$

defined on the space of metrics $\mathcal{M}(M)$ and valued in the space of sections $\Gamma(E)$ of a geometric tensor bundle E over M . These operators are required to be invariant under diffeomorphisms, meaning that

$$F(\Psi_*g) = \Psi_*F(g)$$

for all $g \in \mathcal{M}(M)$ and for all $\Psi \in \text{Diff}(M)$. When M is non-compact and $g \in \mathcal{M}(M)$ is asymptotic to a reference metric g_0 , the further assumption that $F(g_0) = 0$ is made. In the sequel, differential operators that satisfy these properties will be called *curvature operators*. In fact, one may want to consider metrics defined only in a neighborhood of infinity of M . But since every such metric is the restriction of some metric globally defined on M and since we are only interested in its asymptotic behaviour, there is no loss of generality when considering metrics of $\mathcal{M}(M)$ only.

An essential role is then played by the linearization at g_0 of the operator F , particularly by its cokernel

$$\ker DF_{g_0}^* \subset \Gamma(E),$$

where $DF_{g_0}^*$ is the formal L^2 -adjoint of DF_{g_0} . When it is obvious from the context, we shall simply denote it by DF^* . Indeed, given a section $u \in \Gamma(E)$

and a metric $g \in \mathcal{M}(M)$, we expand the contraction of u and $F(g)$ with respect to the reference metric g_0 lifted to $\Gamma(E)$, as

$$\langle u, F(g) \rangle_{g_0} = \langle u, F(g_0) \rangle_{g_0} + \langle u, DF_{g_0}(e) \rangle_{g_0} + \langle u, Q(e) \rangle_{g_0},$$

where $e = g - g_0$, whereas $Q(e) := F(g) - DF_{g_0}(e)$.

In order to involve the formal adjoint $DF_{g_0}^*$, we use the identity

$$\langle u, DF_{g_0}(e) \rangle_{g_0} - \langle DF_{g_0}^*(u), e \rangle_{g_0} = \operatorname{div}^{g_0} \mathbb{U}_F(u, e),$$

where $\mathbb{U}_F(u, e)$ is a 1-form which depends linearly on e and on u . As noted in [36], this 1-form can be chosen as a differential operator of order one less than the order of $F(g)$.

We therefore get

$$\langle u, F(g) \rangle_{g_0} = \operatorname{div}^{g_0} \mathbb{U}_F(u, e) + \langle u, Q(e) \rangle_{g_0}$$

for $u \in \ker DF_{g_0}^*$.

The result obtained by Michel [36] can then be stated as follows.

Theorem 6.2 ([36]). *Let $g \in \mathcal{M}(M)$ and $u \in \ker DF_{g_0}^*$ taken such that the terms $\langle u, F(g) \rangle_{g_0}$ and $\langle u, Q(e) \rangle_{g_0}$ are integrable on a neighborhood of infinity of M . Then the limit*

$$m_F(u, g, g_0) := \lim_{r \rightarrow \infty} \int_{S_r} \mathbb{U}_F(u, e)(\nu_r) dS_r$$

exists, where S_r denotes the geodesic sphere of radius r around a given point $x_0 \in M$, dS_r the induced metric and ν_r the unit normal with respect to g_0 .

Such quantities $m_F(u, g, g_0)$ are called *total charges*. Let us now see them in our context of asymptotically hyperbolic manifolds.

If $(M_0, g_0) = (\mathbb{H}^n, b)$ and if $g \in G_k$, Michel's result also states that the following $O_\uparrow(n, 1)$ -equivariance property holds for total charges,

$$m_F(u, A_*g, g_0) = m_F(A^*u, g, g_0)$$

for all hyperbolic isometries $A \in O_\uparrow(n, 1)$, provided that Michel's Theorem 6.2 holds. Here, as mentioned in [36, Remark 2.6], the group $O_\uparrow(n, 1)$ acts by pushforward on the space $\ker DF_b^*$.

Furthermore, the total charges are under the same assumptions invariant under diffeomorphisms asymptotic to the identity. This is showed in [36, Theorem 3.3]. In our notation, this property is written as

$$m_F(u, (\Psi_0)_*g, b) = m_F(u, g, b)$$

for all $\Psi_0 \in I_0^{k+1}$.

We now combine these results and obtain the following lemma.

Lemma 6.3. *Under the assumptions of Theorem 6.2 with $g \in G_k$, $u \in \ker DF_b^*$, then for all $\Psi \in I^k(g)$, it holds*

$$m_F(u, \Psi_*g, b) = m_F(\pi(\Psi)^*u, g, b),$$

where π denotes again the projection $I^k(g) \rightarrow O_\uparrow(n, 1)$ introduced in Section 2.

We will now link this equivariance property with the linear masses at infinity discussed previously in this paper.

Let $g \in G_k$ and let $V \subset \ker DF_b^*$ be a finite-dimensional subspace which is invariant under the action of $O_\uparrow(n, 1)$. In particular, V is a finite-dimensional representation of this group. We define the functional $H_g \in V^*$ by

$$H_g(u) = m_F(u, g, g_0)$$

for $u \in V$. The above discussion leads to the following corollary.

Corollary 6.4. *Let F be as in Theorem 6.2, and let $V \subset \ker DF_b^*$ be a finite-dimensional subrepresentation of $O_\uparrow(n, 1)$. Then the map $\Phi_F : G_k \rightarrow V^*$ defined by*

$$\Phi_F(g) = H_g$$

is a linear mass at infinity.

Proof. The finite-dimensional space $(\ker DF_b^*)^*$ is naturally a representation of $O_\uparrow(n, 1)$. We just need to check that the intertwining property $\Phi_F(A_*g) = A \cdot \Phi_F(g)$ holds for all $g \in G_k$ and $A \in O_\uparrow(n, 1)$.

We have in fact $A \cdot (\Phi_F(g)) = A \cdot H_g$. The action on $(\ker DF_b^*)^*$ (and therefore on V^*) of $O_\uparrow(n, 1)$ is deduced from the formula

$$A \cdot (\langle H_g, u \rangle) = \langle A \cdot H_g, A \cdot u \rangle = \langle A \cdot H_g, A_*u \rangle.$$

By triviality of the action on \mathbb{R} together with Lemma 6.3, we get

$$\begin{aligned} \langle A \cdot H_g, u \rangle &= \langle H_g, A^*u \rangle \\ &= m_F(A^*u, g, b) \\ &= m_F(u, A_*g, b) \\ &= \langle H_{A_*g}, u \rangle. \end{aligned}$$

Therefore, we have

$$A \cdot H_g = H_{A_*g}$$

as desired. \square

In what follows, we will recall the related Chruściel-Herzlich's definition of the mass based on the scalar curvature operator as in [13]. Section 6.3 introduces a method to construct curvature operators that can be used to construct a given mass from the theory we introduced above. This strategy will be carried out in Sections 6.4 and 6.5. We will in particular discuss carefully the relevant decay assumptions to be made.

6.2. The scalar curvature operator. We first apply this construction with the differential operator $F : \mathcal{M}(\mathbb{H}^n) \rightarrow \mathcal{C}^\infty(\mathbb{H}^n)$ computed from the scalar curvature operator as

$$F(g) := \text{Scal}^g + n(n-1).$$

In [13] (see also [36, Section IV]), the “mass” of an asymptotically hyperbolic metric g is obtained as the linear functional H_g , where the space $\ker D\text{Scal}^*$ is isomorphic to the Minkowski space $\mathbb{R}^{n,1}$. In fact, this space is obtained as

$$\ker D\text{Scal}^* = \left\{ u \in C^\infty(\mathbb{H}^n) \mid \text{Hess}^b u = ub \right\}.$$

It is exactly the space \mathcal{H}_1 of the functions on \mathbb{H}^n obtained as degree 1 homogeneous polynomials of $\mathbb{R}^{n,1}$ evaluated on $\mathbb{H}^n \subset \mathbb{R}^{n,1}$. In particular, all functions f of this space satisfy the growth property

$$u = O(e^r) = O(\rho^{-1}),$$

according to the notation of Section 2 where r is the geodesic distance to the origin of \mathbb{H}^n as measured by b .

This gives indications on the admissible decay properties of the metrics g for which one can assign a mass functional H_g as follows.

Theorem 6.5 (Chruściel-Herzlich, Michel). *Let $\tau > n/2$ and let $g \in G_\tau$ be an asymptotically hyperbolic metric of order τ . Assume moreover that for $e := g - b$, we have $|\nabla e|_b = O(e^{-\tau r})$ and $|\nabla^2 e|_b = O(e^{-\tau r})$, and that $e^r(\text{Scal}_g + n(n-1))$ is integrable in a neighborhood of infinity. Then the mass functional*

$$H_g : u \mapsto \lim_{r \rightarrow \infty} \int_{S_r} \mathbb{U}_{\text{Scal}}(u, e)(\nu_r) dS_r$$

is well defined on the space $\ker D\text{Scal}^*$.

In particular, under Wang’s asymptotics [43], we recover the mass vector formula as in Example 3.9. Indeed, for any $u \in \ker D\text{Scal}^*$ and $g \in G_n$ we have

$$\mathbb{U}_{\text{Scal}}(u, e)(\nu_r) = (nu + \partial_r u) \text{tr}^\sigma m (\sinh r)^{-n} + O(\sinh^{-(n+1)} r)$$

for large r . This is deduced from the general expression of $\mathbb{U}_{\text{Scal}}(u, e)$,

$$\mathbb{U}_{\text{Scal}}(u, e) = u[\text{div } e - d \text{tr } e] - \iota_{\nabla u} e + (\text{tr } e) du.$$

Since $u \in \mathcal{H}_1$ it takes the form $u = \sinh r P(1, x^1, \dots, x^n) + O(1)$, where P is a degree 1 homogeneous polynomial in the variables X^0, X^1, \dots, X^n and where x^i are the standard coordinates on \mathbb{R}^n restricted to the unit sphere \mathbb{S}^{n-1} .

It follows that the integrand we look for is

$$\mathbb{U}_{\text{Scal}}(u, e)(\nu_r) = (n+1)P(1, x^1, \dots, x^n)(\sinh r)^{-(n-1)} \text{tr}^\sigma m + O(e^{-nr}).$$

Taking into account the expression $dS_r = (\sinh r)^{n-1} d\mu^\sigma$, we obtain at the end

$$\int_{S_r} \mathbb{U}_{\text{Scal}}(u, e)(\nu_r) dS_r = (n+1)\Phi_c(m)(P) + o(1),$$

where $\Phi_c : S^2(\mathbb{S}^{n-1}) \rightarrow (\mathbb{R}^{n,1})^*$ is the intertwining map obtained in Theorem 4.5 for the representation $V_{\overline{\omega}_1}$. which also corresponds to Example 3.9.

6.3. Finite dimensional subrepresentations of $C^\infty(\mathbb{H}^n)$ and $S^2(\mathbb{H}^n)$.

The aim of this section is to show that the only finite dimensional subrepresentations of $C^\infty(\mathbb{H}^n)$ and $S^2(\mathbb{H}^n)$ are those we found in Section 4. As a consequence, we shall see that there are infinitely many ways to make those representations be part of the kernel of the adjoint of some linearized curvature operator.

We first study the decomposition of the space of smooth functions on \mathbb{H}^n .

Proposition 6.6. *The space $C^\infty(\mathbb{H}^n)$ decomposes as*

$$C^\infty(\mathbb{H}^n) = \overline{\bigoplus_{p \in \mathbb{N}} \mathcal{H}_p}$$

under the action of $O_\uparrow(n, 1)$. The spaces \mathcal{H}_p are the only finite dimensional irreducible representations of $O_\uparrow(n, 1)$ contained in $C^\infty(\mathbb{H}^n)$.

Proof. The proof is based on a simple argument from algebraic geometry and on approximation.

Let

$$P_{\mathbb{H}^n} := 1 + X^\mu X_\mu$$

and let $\mathfrak{h}_0 = (P_{\mathbb{H}^n})$ be the ideal in $\mathbb{C}[X^0, X^1, \dots, X^n]$ defining \mathbb{H}^n as an algebraic subvariety of $\mathbb{R}^{n,1}$ (note that there is no natural way to enforce the condition $X^0 > 0$ in the definition of the hyperbolic space). Let $A(\mathbb{H}^n) := \mathbb{C}[X^0, X^1, \dots, X^n]/\mathfrak{h}_0$ be the set of regular functions on \mathbb{H}^n . The short exact sequence

$$0 \longrightarrow \mathfrak{h}_0 \longrightarrow \mathbb{C}[X^0, X^1, \dots, X^n] \longrightarrow A(\mathbb{H}^n) \longrightarrow 0,$$

where each term is viewed as an $O_\uparrow(n, 1)$ -representation, splits. Namely, the projection $\mathbb{C}[X^0, X^1, \dots, X^n] \rightarrow A(\mathbb{H}^n)$ induces a vector space isomorphism

$$\bigoplus_{p \in \mathbb{N}} \mathcal{H}_p \simeq A(\mathbb{H}^n).$$

In particular, the last equality gives the decomposition of $A(\mathbb{H}^n)$ into irreducible decompositions. We emphasize here that this reduction is similar in spirit to the standard reduction modulo an ideal (see for example [17]) but it cannot be associated to any monomial ordering. Given $P \in \mathbb{C}[X^0, X^1, \dots, X^n]$, we reduce it modulo $P_{\mathbb{H}^n}$ to get a polynomial H such that $\square_X H = 0$, $H \equiv P \pmod{(P_{\mathbb{H}^n})}$. The algorithm is a sort of Euclidean division based on the decomposition

$$\mathbb{C}_d[X^0, \dots, X^n] = \mathcal{H}_d \oplus (X^\mu X_\mu) \mathbb{C}_{d-2}[X^0, \dots, X^n].$$

We go by downward induction. Assume that all homogeneous components of P of degree greater than $j \geq 0$ are harmonic. Then we can decompose the component $P_{(j)}$ of P that is homogeneous of degree j using the previous decomposition,

$$P_{(j)} = H_{(j)} + (X^\mu X_\mu) U_{(j)},$$

where U_j is homogeneous of degree $j-2$ (note that if $j < 2$ we have $U_{(j)} = 0$). As a consequence, we can replace P by

$$\tilde{P} = P - (1 + X^\mu X_\mu) U_{(j)} \equiv P \pmod{(P_{\mathbb{H}^n})}.$$

The polynomial \tilde{P} has all his homogeneous components of degree $\geq j$ that are wave-harmonic. And we can continue the process down to $j = 0$ to get a polynomial $H \equiv P \pmod{P_{\mathbb{H}^n}}$ such that $\square_X H = 0$.

We now show that if V is any non trivial finite dimensional irreducible subrepresentation of $C^\infty(\mathbb{H}^n)$, then V is one of the \mathcal{H}_p . It follows from the Stone-Weierstrass theorem that $A(\mathbb{H}^n)$ is dense in $C^\infty(\mathbb{H}^n)$ with respect to its Fréchet topology,

$$C^\infty(\mathbb{H}^n) = \overline{\bigoplus_{p \in \mathbb{N}} \mathcal{H}_k}.$$

To prove that there does not exist any other finite dimensional irreducible representation within $C^\infty(\mathbb{H}^n)$, we show that any such representation lies in $A(\mathbb{H}^n)$. Since we have classified the finite dimensional irreducible representations lying in $A(\mathbb{H}^n)$ the conclusion will follow. We set $G := O(n, 1)$ and $H := SO(n) \subset SO(n, 1)$. The hyperbolic space is an homogeneous space

$$\mathbb{H}^n = G/H.$$

As a consequence, we have

$$A(\mathbb{H}^n) \simeq A\left(G/H\right) \simeq A(G)^H,$$

where $A(G)^H$ is the subset of functions of $A(G)$ that are invariant under the right action of H .

Let $\lambda : V \rightarrow C^\infty(\mathbb{H}^n)$ be a G -equivariant linear map from a finite dimensional irreducible representation V of G . Let $\text{ev} : A\left(G/H\right) \rightarrow \mathbb{C}$ be the evaluation at the left coset H . We set $\ell := \text{ev} \circ \lambda$ so ℓ is a linear form on V . For any $h \in H$ and $v \in V$, we have

$$\ell(h \cdot v) = \text{ev}(\lambda(v) \circ h^{-1}) = \text{ev}(\lambda(v)) = \ell(v),$$

so ℓ is H -invariant. The function

$$\lambda(v) : G/H \rightarrow \mathbb{C}$$

satisfies

$$\lambda(v)(gH) = \text{ev}(g^{-1} \cdot \lambda(v)) = \ell(g^{-1} \cdot v).$$

Since V is finite dimensional and G is semi-simple, the representation is algebraic. Hence, $g \mapsto \ell(g^{-1}v) \in A(G)$. By the previous calculation $\lambda(v) \in A(G)^H = A(\mathbb{H}^n)$. So the image of λ is contained in $A(\mathbb{H}^n)$. \square

It can be checked that the elements h of the representation $V_{p\bar{\omega}_1+2\bar{\omega}_2}$ in Lemma A.7 are characterized by the following three conditions

$$\begin{cases} \square h_{\mu\nu} = 0, \\ \eta^{\mu\nu} h_{\mu\nu} = 0, \\ h_{\mu\nu} X^\mu = 0. \end{cases} \quad (43)$$

This has only to be checked for the highest weight vectors. The first two conditions impose h to belong to $A(\mathbb{R}^{n,1}) \otimes \text{Sym}^2(\mathbb{R}^{n,1})$ whose decomposition in irreducible representations is given in Lemma A.7 and the third condition singles out $V_{p\bar{\omega}_1+2\bar{\omega}_2}$ among those representations. We denote by \mathcal{T}_p the set of homogeneous polynomials of degree p that satisfy the conditions (43).

The tangent space of \mathbb{H}^n can be described as an algebraic variety by taking two copies of $A(\mathbb{R}^{n,1})$,

$$\begin{aligned} A(\mathbb{R}^{n,1} \times \mathbb{R}^{n,1}) &= A(\mathbb{R}^{n,1}) \otimes A(\mathbb{R}^{n,1}) \\ &\simeq \mathbb{C}[X^0, X^1, \dots, X^n, Y^0, Y^1, \dots, Y^n], \end{aligned}$$

where we shall call (Y^μ) the second set of variables. These variables should be thought as differentials, that is

$$Y^\mu \simeq dX^\mu.$$

We set

$$T_{\mathbb{H}^n} := Y^\mu X_\mu.$$

The defining ideal \mathfrak{h} for $T\mathbb{H}^n$ is given by

$$\mathfrak{h} = (P_{\mathbb{H}^n}, T_{\mathbb{H}^n}).$$

Note that the first generating polynomial $P_{\mathbb{H}^n}$ is the same as for \mathfrak{h}_0 while the second one, is, up to a factor 2, $DP_{\mathbb{H}^n}(Y)$. The ideal \mathfrak{h} is not homogeneous but is homogeneous for the Y -variables. As a consequence, $A(T\mathbb{H}^n)$ inherits a grading for the Y -variables and we shall denote by $A_d(T\mathbb{H}^n)$ the image of $A_{*,d}(\mathbb{R}^{n,1} \times \mathbb{R}^{n,1}) = A(\mathbb{R}^{n,1}) \otimes \mathbb{C}_d[Y^0, Y^1, \dots, Y^n]$ in $A(T\mathbb{H}^n)$. By polarization, $A_d(T\mathbb{H}^n)$ gets identified with the set of regular sections of $\text{Sym}^d(T\mathbb{H}^n)$.

As in the proof of Proposition 6.6, we find an $O_\uparrow(n, 1)$ -equivariant section of the exact sequence

$$0 \longrightarrow \mathfrak{h}_2 \longrightarrow A_{*,2}(\mathbb{R}^{n,1} \times \mathbb{R}^{n,1}) \longrightarrow A_2(T\mathbb{H}^n) \longrightarrow 0,$$

where $\mathfrak{h}_2 = \{P \in \mathfrak{h} \mid \deg_Y P = 2\}$. We first implement a decomposition of elements in $A_{*,2}(\mathbb{R}^{n,1} \times \mathbb{R}^{n,1})$ into pure trace and trace-free elements. Up to a constant, the trace of P is given by $\square_Y P$ and we can write

$$P = \frac{1}{2(n-1)} (\square_Y P) Y^\mu Y_\mu + \underbrace{\left(P - \frac{1}{2(n-1)} (\square_Y P) Y^\mu Y_\mu \right)}_{\check{P}}$$

for any polynomial $P \in A_{*,2}(\mathbb{R}^{n,1} \times \mathbb{R}^{n,1})$ (by polarization, $Y^\mu Y_\mu$ is the Minkowski metric). The first term is pure trace, while the second one is trace-free meaning that $\square_Y \check{P} = 0$.

As a consequence,

$$A_{*,2}(\mathbb{R}^{n,1} \times \mathbb{R}^{n,1}) = A(\mathbb{R}^{n,1})Y^\mu Y_\mu \oplus \mathring{A}_{*,2}(\mathbb{R}^{n,1} \times \mathbb{R}^{n,1}), \quad (44)$$

where $\mathring{A}_{*,2}(\mathbb{R}^{n,1} \times \mathbb{R}^{n,1})$ denotes the space of trace-free regular functions of degree 2 with respect to Y . We now prove two lemmas that show how the irreducible subrepresentations of each of the terms in Equation (44) descend to the quotient $A_2(T\mathbb{H}^n)$.

Lemma 6.7. *Let $P \in A_{*,2}(\mathbb{R}^{n,1} \times \mathbb{R}^{n,1})$ be a pure trace polynomial, $P = QY^\mu Y_\mu$, with $Q \in \mathbb{R}[X^0, X^1, \dots, X^n]$ such that $\square_X P = 0$. If $P \in \mathfrak{h}$, then $P = 0$.*

Proof. We write $P = QY^\mu Y_\mu$ with $Q \in \mathbb{R}[X^0, X^1, \dots, X^n]$ satisfying $\square_X Q = 0$. Note that the operator ξ^i defined as

$$\xi^i := X^i \frac{\partial}{\partial Y^0} + X^0 \frac{\partial}{\partial Y^i}$$

annihilates both $P_{\mathbb{H}^n}$ and $T_{\mathbb{H}^n}$ so ξ^i maps \mathfrak{h} into itself. However,

$$(\xi^i)^2 P = 2Q((X^0)^2 - (X^i)^2) \notin \mathfrak{h}$$

since Q is not divisible by $P_{\mathbb{H}^n}$. \square

As a consequence of this lemma, the space $\bigoplus_{j \in \mathbb{N}} \mathcal{H}_j Y^\mu Y_\mu$ is mapped injectively into $A_2(T\mathbb{H}^n)$.

The next lemma follows from a straightforward calculation.

Lemma 6.8. *If $\xi \in \Gamma(T\mathbb{R}^{n,1})$ is a vector field on $\mathbb{R}^{n,1}$ and X, Y are vectors tangent to \mathbb{H}^n then*

$$\mathcal{L}_\xi \eta(X, Y) = \mathcal{L}_{\xi^T} b(X, Y) - 2\eta(\nu, \xi)b(X, Y),$$

where, denoting by ν the normal to \mathbb{H}^n , $\xi = \xi^T - \eta(\nu, \xi)\nu$ is the decomposition of ξ into its tangential and normal parts with respect to \mathbb{H}^n .

We are now ready to study the decomposition of the space of smooth symmetric two-tensors on \mathbb{H}^n .

Proposition 6.9. *The space $S^2(\mathbb{H}^n)$ decomposes as*

$$S^2(\mathbb{H}^n) = \overline{\left(\bigoplus_{p \in \mathbb{N}} \mathcal{H}_p b \right)} \oplus \overline{\left(\bigoplus_{p \in \mathbb{N}} \mathcal{T}_p \right)} \oplus \overline{\text{Lie}},$$

under the action of $O_\uparrow(n, 1)$, where Lie denotes the image of

$$\Xi := \left(\bigoplus_{p \in \mathbb{N}, p \geq 2} \Xi_{p\bar{\omega}_1} \right) \oplus \left(\bigoplus_{p \in \mathbb{N}} \Xi_{p\bar{\omega}_1 + \bar{\omega}_2} \right),$$

under the map $\xi \mapsto \mathcal{L}_\xi b$. Here $\Xi_{p\bar{\omega}_1}$ is the image of \mathcal{H}_p under grad^b and $\Xi_{p\bar{\omega}_1 + \bar{\omega}_2}$ is the $U(O_\uparrow(n, 1))$ -module generated by $\xi_{p\bar{\omega}_1 + \bar{\omega}_2}$ (see Equation (64)).

The only finite dimensional irreducible representations of $O_\uparrow(n, 1)$ contained in $S^2(\mathbb{H}^n)$ are the $\mathcal{H}_p b$ and the \mathcal{T}_p and the images of $\Xi_{p\bar{\omega}_1}$ and $\Xi_{p\bar{\omega}_1 + \bar{\omega}_2}$.

Proof. We reuse the notation used in the proof of Proposition 6.6. We have $\mathbb{H}^n = G/H$ and the map $G \mapsto G/H$ shows that G is the oriented orthonormal frame bundle of \mathbb{H}^n . The bundle $\text{Sym}^2(\mathbb{H}^n)$ is the associated vector bundle

$$\mathcal{V} := \text{Sym}^2(\mathbb{H}^n) = G \times_{\mu} \text{Sym}^2(\mathbb{R}^n),$$

where μ is the standard action of H on $V := \text{Sym}^2(\mathbb{R}^n)$. Note that V gets identified with the fiber \mathcal{V}_e above the left coset eH . If $s \in \Gamma(\mathcal{V})$ is a section, we pull it back to a map $\tilde{s} : G \rightarrow V$ by setting

$$\tilde{s}(g) := (g^{-1})_* s(gH) \in \mathcal{V}_e \simeq V.$$

Under this mapping, the space $\Gamma(\mathcal{V})$ gets identified with the following induced representation,

$$\text{Ind}_H^G(V)^\infty := \{f \in C^\infty(G, V) \mid f(gh) = \mu(h^{-1})f(g), \forall g \in G, h \in H\}.$$

The action of G on $\Gamma(\mathcal{V})$ corresponds to the action by left translation on $\text{Ind}_H^G(V)^\infty$. The image of the regular sections $\Gamma_{alg}(\mathcal{V})$ is the induced representation

$$\text{Ind}_H^G(V)^{alg} := \{f \in A(G) \otimes V \mid f(gh) = \mu(h^{-1})f(g), \forall g \in G, h \in H\}.$$

If W is a finite dimensional G -submodule of $\Gamma(\mathcal{V})$ then we can pull it back to $\tilde{W} \subset \text{Ind}_H^G(V)^\infty$. Let $\ell : \tilde{W} \rightarrow V$ be defined by

$$\ell(f) := f(e).$$

The action of G on W is algebraic so $g \mapsto \ell(g^{-1} \cdot f) = f(g)$ is regular. This means that

$$\tilde{W} \subset \text{Ind}_H^G(V)^{alg}$$

and, hence,

$$W \subset \Gamma_{alg}(\mathcal{V}).$$

The Frobenius reciprocity formula [23, Theorem 12.1.3], combined with the branching rules, gives the isotypic decomposition of $\Gamma_{alg}(\mathcal{V})$. Representations with highest weight $p\bar{\omega}_1 + \bar{\omega}_2$ and $p\bar{\omega}_1 + 2\bar{\omega}_2$, $p \in \mathbb{N}$, appear with multiplicity 1 while those with highest weight $p\bar{\omega}_1$, $p \in \mathbb{N}$ appear with multiplicity 2 except for $p = 0$ or 1 for which the multiplicity is 1.

This can be seen as follows. From Lemma 6.7, the space

$$\bigoplus_{p \in \mathbb{N}} \mathcal{H}_p Y^\mu Y_\mu$$

is mapped injectively into $A_2(T\mathbb{H}^n)$. Lemma A.7 gives the decomposition of

$$\mathring{A}_{*,2}(\mathbb{R}^{n,1} \times \mathbb{R}^{n,1}) / (P_{\mathbb{H}^n}) \simeq \left(\bigoplus_{p \in \mathbb{N}} \mathcal{H}_p \right) \otimes \text{Sym}^2(\mathbb{R}^{n,1})$$

into irreducible representations. Quotienting out by $(P_{\mathbb{H}^n}, T_{\mathbb{H}^n})$, we see that $h_{p\bar{\omega}_1 + \bar{\omega}_2}$ becomes proportional to $H_{p\bar{\omega}_1 + \bar{\omega}_2}$. Similarly, $h_{p\bar{\omega}_1}$ and $k_{p\bar{\omega}_1}$ become linear combinations of $H_{p\bar{\omega}_1}$ and $(Z^{-1})^p Y^\mu Y_\mu$. As a consequence, we see that $A_2(T\mathbb{H}^n)$ is generated by the images of \mathcal{T}_p , $\mathcal{H}_p b$, $\Xi_{p\bar{\omega}_1}$ and $\Xi_{p\bar{\omega}_1 + \bar{\omega}_2}$, $p \in \mathbb{N}$.

Finally, we note that vectors in $\Xi_{p\bar{\omega}_1+\bar{\omega}_2}$ are tangent to \mathbb{H}^n while vectors in $\Xi_{p\bar{\omega}_1}$ can be decomposed in a tangential and a normal parts. Using Lemma 6.8, this gives the decomposition of $A_2(T\mathbb{H}^n)$ in the statement of the proposition.

Density of $A_2(T\mathbb{H}^n)$ in $S^2(\mathbb{H}^n)$ is easy to prove. Given h in $S^2(\mathbb{H}^n)$ we can extend it to a section \tilde{h} of $\text{Sym}^2(T\mathbb{R}^{n,1}|_{\mathbb{H}^n})$, where $T\mathbb{R}^{n,1}|_{\mathbb{H}^n}$ denotes the restriction of the tangent space of $\mathbb{R}^{n,1}$ to \mathbb{H}^n . Then we can just approximate each of the $\tilde{h}(\partial_\mu, \partial_\nu)$ in C^∞ by some polynomial and thereby find a polynomial approximation h_0 of \tilde{h} . The image of h_0 in $A_1(T\mathbb{H}^n)$ is then an approximation of h . \square

Lemma 6.10. *Let $F : \mathcal{M}(\mathbb{H}^n) \rightarrow C^\infty(\mathbb{H}^n)$ be a scalar-valued curvature operator vanishing at b and let $DF : S^2(\mathbb{H}^n) \rightarrow C^\infty(\mathbb{H}^n)$ be its linearization at b . Then $DF^*(\mathcal{H}_p)$ is contained in the set*

$$\begin{aligned} & \{\mathcal{L}_{\nabla} f b - 2(p(n-1+p) - (n-1))fb \mid f \in \mathcal{H}_p\} \\ &= \left\{ \text{Hess}^b(f) - (p(n-1+p) - (n-1))fb \mid f \in \mathcal{H}_p \right\} \end{aligned}$$

for $p > 0$, whereas $DF^*(\mathcal{H}_0) = \{0\}$.

Proof. Since F is a curvature operator vanishing at $g = b$, we have

$$0 = F((\Psi_t)_* b)$$

for any 1-parameter family of diffeomorphisms Ψ_t . Therefore, if $\Psi_0 = \text{Id}_{\mathbb{H}^n}$ and if $X = \frac{d}{dt}\Psi_t|_{t=0}$ is the infinitesimal generator of Ψ_t , we have

$$0 = \frac{d}{dt} F((\Psi_t)_* b)|_{t=0} = DF(\mathcal{L}_X b).$$

As a consequence, for any $f \in C^\infty(\mathbb{H}^n)$ and any compactly supported vector field $X \in \Gamma(T\mathbb{H}^n)$ we have

$$\begin{aligned} 0 &= \int_{\mathbb{H}^n} f DF(\mathcal{L}_X b) d\mu^b \\ &= \int_{\mathbb{H}^n} \langle DF^*(f), \mathcal{L}_X b \rangle_b d\mu^b \\ &= -2 \int_{\mathbb{H}^n} \langle \text{div}(DF^*(f)), X \rangle_b d\mu^b, \end{aligned}$$

so we conclude that $\text{div}(DF^*(f)) \equiv 0$. Since DF^* is $O_\uparrow(n, 1)$ -equivariant, we have that the image of \mathcal{H}_p under DF^* is contained in the corresponding isotypic subspace of $A_2(T\mathbb{H}^n)$, see Proposition 6.9. We have thus found that

$$DF^*(f) = \alpha_p f b + \beta_p \mathcal{L}_{\nabla} f b = \alpha_p f b + 2\beta_p \text{Hess}^b(f)$$

for some constants α_p and β_p . The condition $\text{div}(DF^*(f)) = 0$ gives

$$\begin{aligned} 0 &= \alpha_p df + 2\beta_p \text{div Hess}(f) \\ &= \alpha_p df + 2\beta_p (d(\Delta f) + \text{Ric}(df)) \\ &= \alpha_p df + 2\beta_p (p(n-1+p) - (n-1)) df, \end{aligned}$$

see Lemma 6.13. This forces

$$\alpha_p = -2\beta_p(p(n-1+p) - (n-1))$$

unless $p = 0$. In this particular case, $\nabla f = 0$ and the condition $\operatorname{div}(DF^*(f)) = 0$ gives us $\alpha_0 = 0$, thus $DF^*(\mathcal{H}_0) = \{0\}$. \square

As a consequence of this lemma together with Proposition 6.9, Schur's lemma implies the following result, Proposition 6.12, which involves the notion of independent curvature operators.

Definition 6.11. Let $p \in \mathbb{N}$. Two scalar-valued curvature operators $F_i : \mathcal{M}(\mathbb{H}^n) \rightarrow \mathcal{C}(\mathbb{H}^n)$, $i = 1, 2$ are said to be *p-independent* if they are not proportional, and if $DF_1^*(f) \neq DF_2^*(f)$, where f is the highest weight element of the representation \mathcal{H}_p .

Similarly, if $n \geq 4$ and $p \in \mathbb{N}$, we say that two tensor-valued curvature operators $C_i : \mathcal{M}(\mathbb{H}^n) \rightarrow S^2(\mathbb{H}^n)$, $i = 1, 2$ are *p-independent* if they are not proportional, and if $DC_1^*(k) \neq DC_2^*(k)$, where k is the highest weight element of the representation \mathcal{E}_p .

If $n = 3$ and $p \in \mathbb{N}$, the same definition holds, replacing \mathcal{E}_p by the representations $V_{(p+4)\bar{\omega}_1 + p\bar{\omega}_2}$ and $V_{p\bar{\omega}_1 + (p+4)\bar{\omega}_2}$.

Proposition 6.12. *Given two independent scalar-valued curvature operators F_1, F_2 there exists unique coefficients $\lambda_p \in \mathbb{R}$ such that, setting $F_{\lambda_p} := \lambda_p F_1 + (1 - \lambda_p) F_2$,*

$$\mathcal{H}_p \subset \ker DF_{\lambda_p}^*,$$

where $DF_{\lambda_p}^*$ is the L^2 -formal adjoint of DF_{λ_p} .

Similarly, given two Sym^2 -valued independent curvature operators C_1, C_2 there exists unique coefficients $\mu_p \in \mathbb{R}$ such that, setting $C_{\mu_p} := \mu_p C_1 + (1 - \mu_p) C_2$,

$$\mathcal{E}_p \subset \ker DC_{\mu_p}^*,$$

At this point we should notice that Proposition 6.12 can be generalized to linear combinations of an arbitrary number of curvature operators. In particular, we see that there is a priori no preferred way to attach a curvature operator to a given mass. What makes the link between the standard mass and the scalar curvature so tight is the fact that the kernel of the adjoint of the linearization of the scalar curvature is exactly equal to \mathcal{H}_1 and Theorem 6.1. It will become apparent in the next subsection that, in general, $\ker DF^*$ is infinite dimensional for the curvature operators we construct. See however Remark 6.15. There is also to notice that Michel's construction only involves the linearization of curvature operators. This further complicates the possibility of associating one curvature operator to a mass. The link, if it exists, could be provided by positive mass theorems.

6.4. The conformal masses. A key idea is that the space $\ker D\text{Scal}_b^*$, also known as the space of *lapse functions* coincides precisely with the space \mathcal{H}_1 of homogeneous polynomials of degree 1 in the variables X^0, X^1, \dots, X^n , and they are automatically wave-harmonic.

We will consider for $p \geq 0$ the space \mathcal{H}_p of homogeneous wave harmonic polynomials of degree p in the same variables, which was introduced in Section 4.

First, let us give the following result stating that the restriction to a homogeneous solution to the wave equation on $\mathbb{R}^{n,1}$ restricts to an eigenfunction of the Laplacian on \mathbb{H}^n .

Lemma 6.13. *Let $\tilde{u} \in C^\infty(\mathbb{R}^{n+1})$ be a smooth solution to the wave equation*

$$\square \tilde{u} := -\frac{\partial^2 \tilde{u}}{\partial t^2} + \Delta_{\mathbb{R}^n} \tilde{u} = 0,$$

which is moreover homogeneous of degree $p > 0$, $\tilde{u}(sx) = s^p \tilde{u}(x)$ for all $x \in \mathbb{R}^{n+1}$ and $s \in \mathbb{R}$. Then the restriction f of \tilde{u} to the hyperboloid $\mathbb{H}^n \hookrightarrow \mathbb{R}^{n,1}$ solves the equation

$$\Delta u = p(p + n - 1)u,$$

where Δ is the Laplace operator of \mathbb{H}^n .

Proof. In the timelike region the Minkowski metric η on $\mathbb{R}^{n,1}$ can be written as

$$\eta = -ds^2 + s^2 b,$$

where b is the metric on \mathbb{H}^n . Let x a vector in the timelike region of $\mathbb{R}^{n,1}$, then $x = sy$, where $s \in \mathbb{R}$ and $y \in \mathbb{H}^n$. We get

$$\begin{aligned} 0 &= \square \tilde{u}(sy) = \square(s^p \tilde{u}(y)) \\ &= -\frac{1}{s^n} \partial_s(s^n p s^{p-1}) \tilde{u}(y) + \frac{1}{s^2} s^p \Delta \tilde{u} \\ &= s^{p-2} (-p(n-1+p)u(y) + \Delta u(y)). \end{aligned}$$

□

In the light of this Lemma, we look for an operator $F : \mathcal{M}(\mathbb{H}^n) \rightarrow C^\infty(\mathbb{H}^n)$ for which the space $\ker DF_b^*$ contains the space of functions $u \in C^\infty(\mathbb{H}^n)$ such that $\Delta u = p(p + n - 1)u$. It will then contain in particular the finite dimensional representation \mathcal{H}_p of $O_\uparrow(n, 1)$.

As we will see, one such operator F is defined as

$$F_p(g) = \Delta^g \text{Scal}^g - p(p + n - 1) \text{Scal}^g - p(p + n - 1)n(n - 1). \quad (45)$$

As we will see in Subsection 6.3, this is not the only possible choice of a geometric operator F such that \mathcal{H}_p is a subrepresentation of $\ker DF_b^*$. However, we will use it to recover the corresponding linear masses at infinity from Michel's approach of Subsection 6.1. The key element which makes this example work is that the curvature operators ΔScal and $g \mapsto \text{Scal}_g + n(n - 1)$ are independent in the sense of Definition 6.11.

Since we already know properties of the scalar curvature operator from Subsection 6.2, we now have to study the part

$$F_0 : g \mapsto \Delta^g \text{Scal}^g,$$

obtained for $p = 0$. After linearizing it at $g = b$ we get the expressions

$$DF_0^* = D\text{Scal}^* \circ \Delta_b,$$

and

$$\begin{aligned} \mathbb{U}_{F_0}(u, e) = & ud(\text{div div } e) - (\text{div div } e)du - \iota_{\nabla(\Delta u)}e + (\Delta u) \text{div } e \\ & - ud(\Delta(\text{tr } e)) + \Delta(\text{tr } e)du + (\text{tr } e)d(\Delta u) - (\Delta u)d \text{tr } e \\ & + (n-1)(ud(\text{tr } e) - (\text{tr } e)du), \end{aligned}$$

where we have used that $\text{Ric}^b = -(n-1)b$ in the last line. From the results in Subsection 6.2 concerning the scalar curvature operator, the first equation above yields

$$\ker DF_0^* = \{u \in C^\infty(\mathbb{H}^n) \mid \text{Hess}(\Delta u) = (\Delta u)b\},$$

and thus

$$\ker DF_p^* = \left\{ u \in C^\infty(\mathbb{H}^n) \mid \left(\text{Hess}^b - \text{Id } b \right) \left(\Delta^b - p(p+n-1) \right) u = 0 \right\}.$$

We therefore conclude that

$$\mathcal{H}_p \subset \{u \mid \Delta u = p(p+n-1)u\} \subset \ker DF_p^*$$

for $p \geq 0$.

In particular, $\ker DF_p^*$ is infinite-dimensional. We use the first of the above inclusions in order to simplify the expression of $\mathbb{U}_{F_p}(u, e)$ whenever $f \in \mathcal{H}_p$.

By \mathbb{R} -linearity of the expression $\mathbb{U}_F(u, e)$ with respect to its argument F , we indeed have, using $\Delta u - p(p+n-1)u = 0$,

$$\begin{aligned} \mathbb{U}_{F_p}(u, e) = & ud(\text{div div } e) - (\text{div div } e)du - ud(\Delta(\text{tr } e)) \\ & + \Delta(\text{tr } e)du + (n-1)(ud(\text{tr } e) - (\text{tr } e)du). \end{aligned}$$

We now aim at checking that Theorem 6.2 applies to the operators F_p , and that we recover the linear masses at infinity associated to each representation \mathcal{H}_p .

Theorem 6.14. *Let $p \geq 0$. Let $\tau \in \mathbb{R}$ such that $\tau > (p+n-1)/2$ and take $g \in G_\tau$ such that $|\nabla g|_b + |\nabla^2 g|_b + |\nabla^3 g|_b = O(e^{-\tau r})$ and $e^{pr}F_p(g) \in L^1(b)$. Then the map $u \in \mathcal{H}_p \mapsto m_{F_p}(u, g, b)$ given by*

$$m_{F_p}(u, g, b) := \lim_{r \rightarrow +\infty} \int_{S_r} \mathbb{U}_{F_p}(u, e)(\nu_r) dS_r$$

is well defined. Furthermore, the map $H : g \in G_{p+n-1} \mapsto H_g \in (\mathcal{H}_p)^$ defined as*

$$H_g(u) = m_{F_p}(u, g, b)$$

coincides, up to a multiplicative constant, with the linear mass at infinity Φ_c for $n_1 = p$.

Proof. We use the general result of Theorem 6.2 by Michel [36]. Since the growth of elements u of \mathcal{H}_p is $O(e^{pr})$, it suffices to prove that the assumptions imply the convergence of $\langle u, Q_{F_p}(e) \rangle$, where Q_{F_p} is the quadratic remainder

$$Q_{F_p}(e) = F_p(g) - DF_p(e)$$

defined in Subsection 6.1. We have the estimate

$$|Q_{F_p}(e)| \leq C(|e| + |\nabla e| + |\nabla^2 e| + |\nabla^3 e|)^2 \leq Ce^{-2\tau r}.$$

Hence,

$$|\langle u, Q_{F_p}(e) \rangle| = |uQ_{F_p}(e)| \leq C|u|e^{-2\tau r} \leq Ce^{(p-2\tau)r}$$

where we use the fact that $u \in \mathcal{H}_p$. The right hand side is integrable on \mathbb{H}^n with respect to the volume measure $d\mu^b = (\sinh r)^{n-1} dr d\mu^\sigma$ of b , due to the inequality $p - 2\tau < -(n - 1)$. This proves the first part of the theorem.

Then, we first note that the conditions on g imply that it is an element of G_{p+n-1} . We can apply Corollary 6.4 and obtain that the map

$$\Phi_{F_p} : G_{p+n-1} \rightarrow (\mathcal{H}_p)^*$$

is a linear mass at infinity. Hence, from the classification Theorem 4.5, it is proportional to the linear mass at infinity Φ_c obtained for the representation $V_{p\bar{\omega}_1}$.

To be more precise and make sure that the map Φ_{F_p} is not trivial, we compute the expression of $m_{F_p}(u, g, b)$. To do so, we need to analyse the decay of the various terms involved in the expression of $\mathbb{U}_{F_p}(u, e)(\nu_r)$. Recall that by assumption we have

$$e = (\sinh r)^{-(p+n-3)}m + (\sinh r)^{-(p+n-2)}m_1,$$

where the transversality condition holds, $e \in \mathcal{C}^\infty(\mathbb{H}^n \setminus \bar{B}_R, S^2(\mathbb{S}^{n-1}))$. Equivalently, the mass-aspect tensor m of g is a section of $S^2(\mathbb{S}^{n-1})$ while $m_1 \in \mathcal{C}^\infty(\mathbb{H}^n \setminus \bar{B}_R, S^2(\mathbb{S}^{n-1}))$. Note also that the normal vector ν_r is equal to ∂_r in the present geodesic coordinates. We obtain

$$\begin{aligned} \text{tr } e &= (\sinh r)^{-(p+n-3)}(\sinh r)^{-2}(\text{tr}^\sigma m + (\sinh r)^{-1} \text{tr}^\sigma m_1) \\ &= (\sinh r)^{-(p+n-1)} \text{tr}^\sigma m + O(e^{-(p+n)r}), \end{aligned}$$

so

$$\begin{aligned} \Delta(\text{tr } e) &= \partial_r^2(\text{tr } e) + (\sinh r)^{-2} \Delta^\sigma(\text{tr } e) \\ &= (p+n-1)^2(\sinh r)^{-(p+n-1)} \text{tr}^\sigma m + O(e^{-(p+n)r}), \end{aligned}$$

and

$$\text{div div } e = (\sinh r)^{-(p+n+1)} \text{div}^\sigma \text{div}^\sigma m + O(e^{-(p+n+2)r}).$$

One can easily evaluate the ∂_r -derivatives of the above expressions as well, in particular the dominant term keeps the same order of decay. Therefore, we get

$$\begin{aligned}\mathbb{U}_{F_p}(u, e)(\nu_r) &= ((p+n-1)^2 - (n-1)) \cdot \\ &\quad (u(p+n-1) + \partial_r u)(\sinh r)^{-(p+n-1)} \operatorname{tr}^\sigma m + O(e^{-(p+n)r}).\end{aligned}$$

Neither the constant $(p+n-1)^2 - (n-1) \geq (n-1)(n-2)$ nor the factor $(u(p+n-1) + \partial_r u)$ vanish. For the second claim here we write the element $u \in \mathcal{H}_p$ as

$$u = P(X^0, X^1, \dots, X^n) = P(1, x^1, \dots, x^n)(\sinh r)^p + O((\sinh r)^{p-2})$$

where P is a homogeneous polynomial of degree p , and x^i denote the functions on \mathbb{S}^{n-1} obtained as the restriction of the Cartesian coordinates of \mathbb{R}^n . Therefore, we have

$$\partial_r u + (p+n-1)u = P(1, x^1, \dots, x^n)(2p+n-1)(\sinh r)^p + O((\sinh r)^{p-2}).$$

Plugging this into the formula for the integrand $\mathbb{U}_{F_p}(u, e)(\nu_r)$, we have

$$\mathbb{U}_{F_p}(u, e)(\nu_r) = C(n, p)P(1, x^1, \dots, x^n)(\sinh r)^{-(n-1)} + O(e^{-nr}),$$

for a nonzero constant $C(n, p)$. Integrating this against the volume element $dS_r = (\sinh r)^{n-1} d\mu^\sigma$ and taking the limit as $r \rightarrow +\infty$ yields

$$m_{F_p}(u, g, b) = C(n, p) \int_{\mathbb{S}^{n-1}} P(1, x^1, \dots, x^n) \operatorname{tr}^\sigma m d\mu^\sigma.$$

□

Remark 6.15. The interpretation we propose for all the conformal invariants may seem somewhat ad hoc. Namely, we choose the parameter α in Equation (45) so that the kernel of the (formal) adjoint of the linearized operator F_k contains all functions f satisfying $\Delta^b f = d(d+n-1)f$ (perhaps with a certain growth at infinity, because there is to take into consideration the weighted space in which the variation takes place).

To show that this construction is more than this, we do some kind of reverse engineering and take a closer look at the mass Φ_c with $n_1 = 2$. We are looking for some curvature operator \mathcal{C} such that the kernel of the adjoint of \mathcal{C} linearized at $g = b$ is exactly \mathcal{H}_2 .

The functions $V \in \mathcal{H}_2$ are polynomials of degree 2 and, hence, are characterized by $\partial^{(3)} V \equiv 0$ on $\mathbb{R}^{n,1}$. The condition $\square V = 0$ is in a sense a maximal growth condition since a complementary subspace of \mathcal{H}_2 in $\mathbb{R}_2[X^m u]$ is generated by $(X^0)^2 - \sum_i (X^i)^2$ which is constant on \mathbb{H}^n and, hence, should give 0 when inserted in \mathbb{U}_2 .

The equation satisfied by the restriction of V 's to \mathbb{H}^n is given by

$$\mathcal{C}_0^*(V) := \nabla_i \nabla_j \nabla_k V = 2b_{jk} \nabla_i V + b_{ij} \nabla_k V + b_{ik} \nabla_j V. \quad (46)$$

Since this is a third order operator, there is no chance for it to be the adjoint of a certain linearized operator depending on the metric g only.

However, the Christoffel symbols of the metric g have three free indices. This indicates that we should linearize the curvature operator \mathcal{C} with respect to the connection and the metric, that is adopt a point of view à la Palatini (see for example [42, Appendix E]).

In what follows, we shall work only with the higher order terms, that is at the level of the principal symbol to keep matters simple. This serves just as a motivation for the introduction of the curvature operator F_k .

Let ∇^b denote the Levi-Civita connection of the hyperbolic metric and let ∇^g denote the Levi-Civita connection of an arbitrary metric g that coincides with b outside a compact set. Let $\Gamma := \nabla^g - \nabla^b$ be the Christoffel symbols of the connection ∇^g with respect to the background metric b ,

$$\Gamma_{ij}^k = \frac{1}{2} b^{kl} \left(\nabla_i^b g_{lj} + \nabla_j^b g_{il} - \nabla_l^b g_{ij} \right).$$

The minimal coupling between Γ and $\nabla^{(3)}$ is given by

$$\mathcal{C}_0 = \nabla^i \nabla^j \nabla_k \Gamma_{ij}^k + \{\text{lower order terms}\},$$

where indices are raised and lowered with respect to either g or b . This does not matter since we are looking at the first order variation of \mathcal{C} around $g = b$, $\Gamma = 0$. One then recognizes the first term in the definition of (the double divergence of) the Ricci tensor. If, instead of considering Equation (46), we use the following equivalent form

$$\mathcal{C}^*(V) := \nabla_i \nabla_j \nabla_k V - \nabla_i \Delta V b_{jk} = -2n \nabla_i V b_{jk} + b_{ij} \nabla_k V + b_{ik} \nabla_j V, \quad (47)$$

we find exactly that \mathcal{C} is (at the level of the principal symbol) the double divergence of the Ricci curvature. By the second Bianchi identity, we have that $2 \operatorname{div} \operatorname{div} \operatorname{Ric} = \Delta \operatorname{Scal}$. This justifies the introduction of the curvature operator F_k at least for \mathcal{H}_2 .

This construction raises the following question: is there a formalism in Riemannian geometry that allows one to vary jets of arbitrary orders independently? That is to say do there exist higher order Palatini formalisms? That would allow to generalize the previous method to find natural candidates for the curvature operator \mathcal{C}_k associated to all \mathcal{H}_k . Note that even if such a formalism exists, it would require controlling some weighted $W^{k,2}$ -norm of $g - b$ and the weighted L^1 -norm of \mathcal{C}_k . The curvature operator F_k serves the same purpose with less derivatives of $g - b$ to control.

6.5. The Weyl masses. In this subsection, motivated by Proposition 6.12, we aim at finding two independent (in the sense of Definition 6.11) $S^2(\mathbb{H}^n)$ -valued curvature operators C_1 and C_2 and, for each $p \in \mathbb{N}$, a linear combination of them, $C_{\mu_p} := \mu_p C_1 + (1 - \mu_p) C_2$, such that the representation \mathcal{E}_p is a subset of $\ker DC_{\mu_p}^*$. This in the case $n \geq 4$.

For the case $n = 3$ we wish to find, for each $p \in \mathbb{N}$, such combinations for which the representation $V_{(p+4)\overline{\omega}_1 + p\overline{\omega}_2}$ (resp. $V_{p\overline{\omega}_1 + (p+4)\overline{\omega}_2}$) is a subset of $\ker DC_{\mu_p}^*$.

We obtain the following two results, treating separately the cases $n \geq 4$ and $n = 3$.

Proposition 6.16. *For $n \geq 4$ and $p \in \mathbb{N}$, the curvature operators $C_1 : g \mapsto \text{Ric}_g + (n-1)g$ and $C_2 = \mathcal{B}$, the Bach tensor operator, are independent. In particular, Proposition 6.12 applies.*

Proposition 6.17. *For $n = 3$ and $p \in \mathbb{N}$, the curvature operators $C_1 : g \mapsto \text{Ric}_g + 2g$ and $C_2 = \mathcal{C}$, the Cotton-York tensor operator, are independent. In particular, Proposition 6.12 applies.*

The proof of the propositions requires us to make sure that the formal adjoints of the linearization at b of the chosen operators C_1 and C_2 are different when evaluated at the highest weight elements of the relevant representations.

More precisely, we will check that, for a general k belonging to the relevant representation space (either \mathcal{E}_p for Proposition 6.16 or $V_{(p+4)\bar{\omega}_1 + p\bar{\omega}_2}$ or $V_{p\bar{\omega}_1 + (p+4)\bar{\omega}_2}$ for Proposition 6.17), the images $DC_1^*(k)$ and $DC_2^*(k)$ are proportional to k : $DC_i^*(k) = p_i k$, with $p_1 \neq p_2$.

We begin with the computation of $DRic_b^*(k)$, for $k \in \mathcal{E}_p$.

6.5.1. *Calculations for the Ricci tensor.* Let h be an arbitrary Lorentzian metric on $\mathbb{R}^{n,1}$ such that \mathbb{H}^n is a spacelike hypersurface.

For any vectors X, Y tangent to \mathbb{H}^n , we set

$$S(X, Y) := h(X, {}^h\nabla_Y \nu), \quad (48)$$

where ν is the unit future pointing normal to \mathbb{H}^n . Unit meaning here that $h(\nu, \nu) \equiv -1$. We extend ν to a neighborhood of \mathbb{H}^n so that it satisfies the geodesic equation ${}^h\nabla_\nu \nu = 0$. Let g denote the metric induced by h on \mathbb{H}^n .

The following calculations are standard and can be found for example in [12, Chapter 6, Section 4]. The Levi-Civita connection ${}^g\nabla$ of g is the restriction of ${}^h\nabla$ to $T\mathbb{H}^n$,

$${}^g\nabla_X Y = {}^h\nabla_X Y - S(X, Y)\nu.$$

It follows that, if X, Y, Z are vector fields tangent to \mathbb{H}^n we have

$$\begin{aligned} {}^g\mathcal{R}(X, Y)Z &= {}^h\mathcal{R}(X, Y)Z + S(X, Z)S(Y) - S(Y, Z)S(X) \\ &\quad - ({}^g\nabla_X S(Y, Z) - {}^g\nabla_Y S(X, Z))\nu, \end{aligned}$$

and

$${}^h\mathcal{R}(X, \nu)\nu = -\left({}^h\nabla_\nu S(X) + S(S(X))\right)\nu,$$

where $S(X)$ denotes the shape operator,

$$S(X) := {}^h\nabla_X \nu.$$

The tangential components of the Ricci tensor of h can be computed as follows. Let (e_1, \dots, e_n) be an orthonorma frame on \mathbb{H}^n . Then

$$\begin{aligned}
{}^h\text{Ric}(X, Y) &= \sum_{i=1}^n h \left(Y, {}^h\mathcal{R}(X, e_i)e_i \right) + h \left(Y, {}^h\mathcal{R}(X, \nu)\nu \right) \\
&= \sum_{i=1}^n h \left(Y, {}^g\mathcal{R}(X, e_i)e_i - S(X, e_i)S(e_i) + S(e_i, e_i)S(X) \right) \\
&\quad - h \left(Y, \left({}^h\nabla_\nu S(X) + S(S(X)) \right) \right) \\
&= {}^g\text{Ric}(X, Y) + HS(X, Y) - 2g(S(X), S(Y)) - h \left(Y, {}^h\nabla_\nu S(X) \right) \\
&= {}^g\text{Ric}(X, Y) + HS(X, Y) - 2g(S(X), S(Y)) - {}^h\nabla_\nu S(X, Y),
\end{aligned}$$

where $H = \sum_i S(e_i, e_i) = {}^h \text{div}(\nu)$ is the mean curvature.

Let now $h = \eta + tk + O(t^2)$ be a 1-parameter family of metrics such that $k(\nu, \cdot) = 0$ on $\mathbb{R}^{n,1}$. Then at first order, ν remains the normal to \mathbb{H}^n and for arbitrary vector fields U and V ,

$$\begin{aligned}
\frac{d}{dt} {}^h\nabla_U V|_{t=0} &= U^\alpha V^\beta \frac{d}{dt} {}^h\Gamma_{\alpha\beta}^\gamma \partial_\gamma|_{t=0} \\
&= \frac{1}{2} U^\alpha V^\beta \eta^{\gamma\delta} \left(\eta \nabla_\alpha k_{\delta\beta} + \eta \nabla_\beta k_{\alpha\delta} - \eta \nabla_\delta k_{\alpha\beta} \right) \partial_\gamma.
\end{aligned}$$

In particular,

$$\begin{aligned}
\eta \left(X, \frac{d}{dt} S(Y)|_{t=0} \right) &= \eta \left(X, \frac{d}{dt} \nabla_Y \nu|_{t=0} \right) \\
&= \frac{1}{2} X^\delta Y^\alpha \nu^\beta \left(\eta \nabla_\alpha k_{\delta\beta} + \eta \nabla_\beta k_{\alpha\delta} - \eta \nabla_\delta k_{\alpha\beta} \right) \\
&= \frac{1}{2} \nabla_\nu k(X, Y) - \frac{1}{2} (k(X, S(Y)) - k(Y, S(X))) \\
&= \frac{1}{2} \nabla_\nu k(X, Y),
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt} S(X, Y)|_{t=0} &= \frac{d}{dt} (h(X, S(Y)))|_{t=0} \\
&= k(X, S(Y)) + \eta \left(X, \frac{d}{dt} S(Y) \right) \\
&= k(X, S(Y)) + \frac{1}{2} \eta \nabla_\nu k(X, Y),
\end{aligned}$$

where we used the fact that $k(\nu, \cdot) = 0$ and

$${}^\eta S(X) = \frac{1}{\sqrt{-x^\alpha x_\alpha}} X + \frac{x_\alpha X^\alpha x^\mu \partial_\mu}{(-x^\alpha x_\alpha)^{3/2}}.$$

Assuming that k is tracefree, $\eta^{\mu\nu}k_{\mu\nu} = 0$, we have

$$\frac{d}{dt}\sqrt{-\det(h)}|_{t=0} = \frac{-\eta^{\mu\nu}k_{\mu\nu}}{2\sqrt{-\det(\eta)}} = 0.$$

As a consequence

$$\begin{aligned} \frac{dH}{dt}|_{t=0} &= \frac{d}{dt}h \operatorname{div}(\nu) \\ &= \frac{d}{dt} \left(\frac{1}{\sqrt{-\det(h)}} \partial_\mu \left(\sqrt{-\det(h)} \nu^\mu \right) \right) \\ &= 0. \end{aligned}$$

Before computing the time derivative of ${}^h\operatorname{Ric}(X, Y)$, we compute the (a priori) most complicated term. In the following computation all the terms are evaluated at point of \mathbb{H}^n ,

$$\begin{aligned} \frac{d}{dt}h \left(Y, {}^h\nabla_\nu S(X) \right) |_{t=0} &= k(Y, {}^\eta\nabla_\nu S(X)) + \eta \left(Y, \frac{d}{dt} {}^h\nabla_\nu S(X) \right) \\ &= -k(Y, S(S(X))) + Y_\alpha \frac{d}{dt} \left({}^h\nabla_\nu S^\alpha_\beta \right) X^\beta \\ &= -k(Y, X) + Y_\alpha \nu^\gamma \frac{d}{dt} \left(\partial_\gamma S^\alpha_\beta + \Gamma_{\gamma\delta}^\alpha S^\delta_\beta - \Gamma_{\gamma\beta}^\delta S^\alpha_\delta \right) X^\beta \\ &= -k(Y, X) + Y_\alpha \nu^\gamma \left(\eta \nabla_\gamma \frac{d}{dt} S^\alpha_\beta + \left(\frac{d}{dt} \Gamma \right)_{\gamma\delta}^\alpha S^\delta_\beta - \left(\frac{d}{dt} \Gamma \right)_{\gamma\beta}^\delta S^\alpha_\delta \right) X^\beta \\ &= -k(Y, X) + Y_\alpha \nu^\gamma \left(\eta \nabla_\gamma \frac{d}{dt} S^\alpha_\beta + \left(\frac{d}{dt} \Gamma \right)_{\gamma\beta}^\alpha - \left(\frac{d}{dt} \Gamma \right)_{\gamma\beta}^\alpha \right) X^\beta \\ &= -k(Y, X) + Y_\alpha \nu^\gamma X^\beta \eta \nabla_\gamma \frac{d}{dt} S^\alpha_\beta \\ &= -k(Y, X) + \frac{1}{2} \eta \nabla_\nu {}^\eta\nabla_\nu k(X, Y). \end{aligned}$$

Here we used

$$\begin{aligned} \frac{d}{dt}S^\alpha_\beta|_{t=0} &= \frac{d}{dt}(h^{\alpha\mu}S_{\mu\beta}) \\ &= -k^{\alpha\mu}\eta_{\mu\beta} + \eta^{\alpha\mu} \left(k_{\mu\beta} + \frac{1}{2} \eta \nabla_\nu k_{\mu\beta} \right) \\ &= \frac{1}{2} \eta \nabla_\nu k^\alpha_\beta. \end{aligned}$$

The derivative of ${}^h\operatorname{Ric}$ (restricted to the tangent space of \mathbb{H}^n) can then be obtained as

$$\begin{aligned} \frac{d}{dt}{}^h\operatorname{Ric}(X, Y)|_{t=0} &= D\operatorname{Ric}_b(k)(X, Y) - \frac{1}{2} \eta \nabla_\nu {}^\eta\nabla_\nu k(X, Y) \\ &\quad + \frac{n-4}{2} \eta \nabla_\nu k(X, Y) + (n-1)k(X, Y). \end{aligned}$$

Assuming further that k is a solution to the linearized Einstein equations,

$$\frac{d}{dt} {}^h \text{Ric} = 0,$$

we obtain

$$D\text{Ric}_b(k) = -(n-1)k - \frac{n-4}{2} {}^\eta \nabla_\nu k + \frac{1}{2} {}^\eta \nabla_\nu {}^\eta \nabla_\nu k. \quad (49)$$

If k is homogeneous of degree p , then ${}^\eta \nabla_\nu k = pk$ and ${}^\eta \nabla_\nu {}^\eta \nabla_\nu k = p(p-1)k$, so Equation (49) becomes

$$D\text{Ric}_b(k) = - \left(n-1 + \frac{n-4}{2}p - \frac{p(p-1)}{2} \right) k. \quad (50)$$

Note that the Ricci tensor operator is formally self-adjoint; indeed, it is the case of the Einstein tensor operator since it is obtained as the first variation of the Einstein-Hilbert functional. Hence we find that for $p = 2$, for any $n \geq 3$, such tensors k belong to the kernel of $D\text{Ric}_b^*$.

Let C_1 be the operator defined by

$$C_1(g) := \text{Ric}_g + (n-1)g.$$

It is a curvature operator in the sense of Proposition 6.12, in particular it vanishes at $g = b$. From (50), it satisfies

$$DC_1^*(k) = \frac{1}{2}(p^2 - (n-3)p)k, \quad (51)$$

for homogeneous polynomial tensors k of degree p such as above.

6.5.2. Calculations for the Cotton-York tensor. In this subsection we assume that $n = 3$. The Cotton-York tensor of a Riemannian metric g is defined as

$$\mathcal{C}_{ij} := \varepsilon_i^{kl} \nabla_k \mathcal{P}_{lj},$$

where $\mathcal{P} := \text{Ric} - \frac{1}{4} \text{Scal} g$ is the Schouten tensor of the metric g .

Given a metric g on \mathbb{H}^3 , we extend it to the restriction of the tangent bundle $T\mathbb{R}^{3,1}$ to \mathbb{H}^3 by setting $g(\nu, \cdot) = \eta(\nu, \cdot)$. We also extend the Schouten tensor of g to $T\mathbb{R}^{3,1}$ by setting $\mathcal{P}(\nu, \cdot) = 0$.

We can also extend the Cotton-York tensor trivially in the ν direction. This corresponds to the following formula for \mathcal{C} :

$$\mathcal{C}_{\alpha\beta} := \varepsilon_{\alpha\gamma\delta\iota} g^{\gamma\iota} \nabla^\gamma \mathcal{P}_\beta^\delta \nu^\iota. \quad (52)$$

With this way of extending \mathcal{P} to $T\mathbb{R}^{3,1}$ it makes no difference to compute the covariant derivative with respect to the metric g or with respect to the metric h since the difference only involves $\mathcal{P}(\nu, \cdot)$.

We now return to the notations of the previous section. g and h are 1-parameter families of metrics with $h = \eta + tk + O(t^2)$ and $g = h|_{T\mathbb{H}^3}$. We

have

$$\begin{aligned}
\frac{d}{dt}\mathcal{C}_{\alpha\beta} &= \varepsilon_{\alpha\gamma\delta\iota}\nu^\iota \frac{d}{dt} \left({}^g\nabla^{\gamma'}\mathcal{P}_{\delta'}^\beta \right) \\
&= \varepsilon_{\alpha\gamma\delta\iota}\nu^\iota \frac{d}{dt} \left(h^{\gamma\gamma'} h^{\delta\delta'} {}^g\nabla_{\gamma'}\mathcal{P}_{\delta'\beta} \right) \\
&= \varepsilon_{\alpha\gamma\delta\iota}\nu^\iota h^{\gamma\gamma'} h^{\delta\delta'} \frac{d}{dt} \left({}^g\nabla_{\gamma'}\mathcal{P}_{\delta'\beta} \right) \\
&= \varepsilon_{\alpha\gamma\delta\iota}\nu^\iota h^{\gamma\gamma'} h^{\delta\delta'} \frac{d}{dt} \left({}^h\nabla_{\gamma'}\mathcal{P}_{\delta'\beta} \right),
\end{aligned}$$

since ${}^g\nabla\mathcal{P} = 0$ at $t = 0$.

We now compute $\frac{d}{dt}({}^h\nabla_{\gamma'}\mathcal{P}_{\delta'\beta})$. Note that for $t = 0$ we have $\mathcal{P} = {}^b\mathcal{P} = -\frac{1}{2}b$ and $b_{\alpha\beta} = \eta_{\alpha\beta} + x_\alpha x_\beta$.

$$\begin{aligned}
&\frac{d}{dt} \left({}^h\nabla_{\gamma'}\mathcal{P}_{\delta'\beta} \right) \\
&= \frac{d}{dt} \left(\partial_{\gamma'}\mathcal{P}_{\delta'\beta} - \Gamma_{\gamma'\delta'}^\kappa \mathcal{P}_{\kappa\beta} - \Gamma_{\gamma'\beta}^\kappa \mathcal{P}_{\delta'\kappa} \right) \\
&= \partial_{\gamma'} \left(\frac{d}{dt}\mathcal{P}_{\delta'\beta} \right) - \left(\frac{d}{dt}\Gamma \right)_{\gamma'\delta'}^\kappa \mathcal{P}_{\kappa\beta} - \left(\frac{d}{dt}\Gamma \right)_{\gamma'\beta}^\kappa \mathcal{P}_{\delta'\kappa} \\
&= \partial_{\gamma'} \left(\frac{d}{dt}\mathcal{P}_{\delta'\beta} \right) + \frac{1}{2} \left(\frac{d}{dt}\Gamma \right)_{\gamma'\delta'}^\kappa b_{\kappa\beta} + \frac{1}{2} \left(\frac{d}{dt}\Gamma \right)_{\gamma'\beta}^\kappa b_{\delta'\kappa} \\
&= \partial_{\gamma'} \left(\frac{d}{dt}\mathcal{P}_{\delta'\beta} \right) + \frac{1}{4}\eta^{\kappa\tau} \left(\partial_{\gamma'}k_{\tau\delta'} + \partial_{\delta'}k_{\gamma'\tau} - \partial_\tau k_{\gamma'\delta'} \right) b_{\kappa\beta} \\
&\quad + \frac{1}{4}\eta^{\kappa\tau} \left(\partial_{\gamma'}k_{\tau\beta} + \partial_\beta k_{\gamma'\tau} - \partial_\tau k_{\gamma'\beta} \right) b_{\delta'\kappa} \\
&= \partial_{\gamma'} \left(\frac{d}{dt}\mathcal{P}_{\delta'\beta} \right) + \frac{1}{2}\partial_{\gamma'}k_{\beta\delta'} \\
&\quad - \frac{1}{2} \left(x_\beta k_{\gamma'\delta'} + x_{\delta'} k_{\beta\gamma'} \right) + \frac{1}{4} \left(x_\beta x^\tau \partial_\tau k_{\gamma'\delta'} + x_{\delta'} x^\tau \partial_\tau k_{\beta\gamma'} \right) \\
&= \partial_{\gamma'} \left(\frac{d}{dt}\mathcal{P}_{\delta'\beta} \right) + \frac{1}{2}\partial_{\gamma'}k_{\beta\delta'} + \frac{p-2}{4} \left(x_\beta k_{\gamma'\delta'} + x_{\delta'} k_{\beta\gamma'} \right).
\end{aligned} \tag{53}$$

Note that for the hyperbolic space $\nu^\iota = x^\iota$. Hence,

$$\begin{aligned}
\frac{d}{dt}\mathcal{C}_{\alpha\beta} &= \varepsilon_{\alpha\gamma\delta\iota}\nu^\iota \eta^{\gamma\gamma'} \eta^{\delta\delta'} \frac{d}{dt} \left({}^h\nabla_{\gamma'}\mathcal{P}_{\delta'\beta} \right) \\
&= \varepsilon_{\alpha\gamma\delta\iota}\nu^\iota \eta^{\gamma\gamma'} \eta^{\delta\delta'} \left(\partial_{\gamma'} \left(\frac{d}{dt}\mathcal{P}_{\delta'\beta} \right) + \frac{1}{2}\partial_{\gamma'}k_{\beta\delta'} \right).
\end{aligned}$$

Now note that

$$\frac{d}{dt}\mathcal{P}_{\delta'\beta} = \frac{d}{dt}\text{Ric}_{\delta'\beta} = -k_{\delta'\beta},$$

so

$$\frac{d}{dt}\mathcal{C}_{\alpha\beta} = -\frac{1}{2}\varepsilon_{\alpha\gamma\delta\iota}\nu^\iota \eta^{\gamma\gamma'} \eta^{\delta\delta'} \partial_{\gamma'}k_{\delta'\beta}. \tag{54}$$

One can now plug $k = H_{p\bar{\omega}_1+(p+4)\bar{\omega}_2}$ or $k = H_{(p+4)\bar{\omega}_1+p\bar{\omega}_2}$ into (54) and get that

$$\begin{aligned} DC_b(H_{p\bar{\omega}_1+(p+4)\bar{\omega}_2}) &= -\frac{i(p+3)}{2}H_{p\bar{\omega}_1+(p+4)\bar{\omega}_2}, \\ DC_b(H_{(p+4)\bar{\omega}_1+p\bar{\omega}_2}) &= \frac{i(p+3)}{2}H_{(p+4)\bar{\omega}_1+p\bar{\omega}_2}. \end{aligned}$$

Note that the Cotton-York tensor \mathcal{C} is formally self-adjoint (it is indeed the first variation of the Chern-Simons functional, see [38]), hence the $(2,0)$ -tensor $DC_b^*(k)$ coincides with $DC_b(k)$.

Hence, for $k = H_{p\bar{\omega}_1+(p+4)\bar{\omega}_2}$, in particular k homogeneous of degree p , we find that

$$DC_2^*(k) = p_2 k, \quad p_2 = -\frac{i(p+3)}{2},$$

where the curvature operator $C_2 = \mathcal{C}$ is the Cotton-York tensor.

On the other hand, we obtained from (51)

$$DC_1^*(k) = p_1 k, \quad p_1 = \frac{1}{2}p^2.$$

Since $p_2 \neq p_1$, there is a unique choice of a constant $\mu_p \in \mathbb{R}$, namely $\mu_p = -\frac{p_2}{p_2-p_1}$, such that $DC_{\mu_p}^*(k) = 0$, where $C_{\mu_p} = \mu_p C_1 + (1-\mu_p)C_2$. This forces the representation $V_{p\bar{\omega}_1+(p+4)\bar{\omega}_2}$ to be included in $\ker DC_{\mu_p}^*$.

The same occurs for the representation $V_{(p+4)\bar{\omega}_1+p\bar{\omega}_2}$ when one takes $p_2 = +\frac{i(p+3)}{2}$ instead, and this concludes the proof of Proposition 6.17.

6.5.3. Calculations for the Bach tensor. The Bach tensor is the variation of the quadratic functional $\int |W|^2 d\mu^g$. As such its linearization is a formally self adjoint operator. See [8, Chapter 4 H, Paragraph 4.76]. Up to an irrelevant multiplicative constant it is given by

$$\mathcal{B}_{ab} := \mathcal{P}^{ij} W_{iajb} - \nabla^c (\nabla_c \mathcal{P}_{ab} - \nabla_a \mathcal{P}_{cb}).$$

We are going to compute $\frac{d}{dt} g \mathcal{B}_{ab}$ where g is the 1-parameter family of metrics given above. As before, we assume that all tensors defined on \mathbb{H}^n are extended to $\mathbb{R}^{n,1}$ trivially in the ν -direction. The first point to notice is that ${}^b\mathcal{P} = \frac{n+2}{2}b$ and ${}^bW = 0$ so

$$\frac{d}{dt}|_{t=0} \mathcal{P}^{ij} W_{iajb} = 0$$

and

$$\frac{d}{dt}|_{t=0} \nabla^c (\nabla_c \mathcal{P}_{ab} - \nabla_a \mathcal{P}_{cb}) = \nabla^c \frac{d}{dt}|_{t=0} ({}^g\nabla_c \mathcal{P}_{ab} - {}^g\nabla_a \mathcal{P}_{cb}).$$

Since \mathcal{P} was extended trivially to $T\mathbb{R}^{n,1}$, we can replace the covariant derivative ${}^g\mathcal{P}$ by ${}^h\mathcal{P}$ as long as we evaluate it in directions tangential to \mathbb{H}^n . As a consequence,

$$\frac{d}{dt}|_{t=0} ({}^g\nabla_c \mathcal{P}_{ab} - {}^g\nabla_a \mathcal{P}_{cb})$$

is given by Equation (53),

$$\begin{aligned} T_{\gamma\alpha\beta} &:= \frac{d}{dt}|_{t=0} ({}^g\nabla_\gamma \mathcal{P}_{\alpha\beta} - {}^g\nabla_\alpha \mathcal{P}_{\gamma\beta}) \\ &= - \left(n - \frac{5}{2} + \frac{n-3}{2}p \right) (\partial_\gamma k_{\alpha\beta} - \partial_\alpha k_{\gamma\beta}) + \frac{p-2}{4} (x_\alpha k_{\gamma\beta} - x_\gamma k_{\alpha\beta}). \end{aligned}$$

Simple calculations show that

$$\begin{aligned} x^\gamma T_{\gamma\alpha\beta} &= \left[\frac{p-2}{4} - \left(n - \frac{5}{2} + \frac{n-3}{2}p \right) (p+1) \right] k_{\alpha\beta}, \\ x^\beta T_{\gamma\alpha\beta} &= 0. \end{aligned}$$

We project T to a tensor U so that it is zero when contracted with x in any slot and so that $U(X, Y, Z) = T(X, Y, Z)$ for X, Y, Z tangent to \mathbb{H}^n . It can be checked that the tensor

$$U_{\gamma\alpha\beta} := - \left(n - \frac{5}{2} + \frac{n-3}{2}p \right) [\partial_\gamma k_{\alpha\beta} - \partial_\alpha k_{\gamma\beta} + (p+1)(x_\gamma k_{\alpha\beta} - x_\alpha k_{\gamma\beta})]$$

satisfies these conditions. We can finally complete the calculation of the variation of the Bach tensor of g ,

$$\begin{aligned} \frac{d}{dt}|_{t=0} \mathcal{B}_{\alpha\beta} &= -b^{\gamma\delta b} \nabla_\delta \frac{d}{dt}|_{t=0} ({}^g\nabla_\gamma \mathcal{P}_{\alpha\beta} - {}^g\nabla_\alpha \mathcal{P}_{\gamma\beta}) \\ &= -b^{\gamma\delta} \partial_\delta U_{\gamma\alpha\beta} \\ &= -n(p+1) \left(n - \frac{5}{2} + \frac{n-3}{2}p \right) k_{\alpha\beta}. \end{aligned}$$

From the formally self-adjoint property already mentioned concerning the linearized Bach tensor, we also obtain

$$D\mathcal{B}^*(k) = -n(p+1) \left(n - \frac{5}{2} + \frac{n-3}{2}p \right) k.$$

for $k \in \mathcal{E}_p$.

Summarizing the results for $n \geq 4$, we have two curvature operators $C_1 : g \mapsto \text{Ric}_g + (n-1)g$ and $C_2 = \mathcal{B}$ with $DC_i^*(k) = p_i k$, where the numbers

$$\begin{aligned} p_1 &= \frac{1}{2}(p^2 - (n-3)p) = \frac{1}{2}p(p-n+3), \\ p_2 &= -n(p+1) \left(n - \frac{5}{2} + \frac{n-3}{2}p \right) \end{aligned}$$

are different for all $n \geq 4$, $p \in \mathbb{N}$. Hence C_1 and C_2 are independent, thus proving Proposition 6.16: there exists a unique constant $\mu_p \in \mathbb{R}$ such that the curvature operator $C_{\mu_p} = \mu_p C_1 + (1-\mu_p)C_2$ is such that $\mathcal{E}_p \subset \ker DC_{\mu_p}^*$.

Remark 6.18. In principle, we could compute for each such curvature operator C_{μ_p} the associated linear mass $\Phi_{C_{\mu_p}}$ obtained from Corollary 6.4.

We leave for a future work the explicit computation of the coefficient relating this map and the corresponding map obtained in the classification Theorem 4.5.

APPENDIX A. ON THE IRREDUCIBLE REPRESENTATIONS THAT APPEAR IN THE CLASSIFICATION

An important fact concerning the (complex) representations that appear in Theorem 4.5 is that they are of real type, namely they are the complexification of real irreducible representations of $O_{\uparrow}(n, 1)$. See for example [20, Section 26.3] for an introduction to real and quaternionic representations. The situation is slightly more complicated in dimension 3 for the Weyl masses since one has to consider $V_{n_1\bar{\omega}_1+(n_1+4)\bar{\omega}_2} \oplus V_{(n_1+4)\bar{\omega}_1+n_1\bar{\omega}_2}$ to get a real representation.

The real representations whose complexifications give the representations \mathcal{H}_{n_1} and \mathcal{W}_{n_1} are easy to find. Namely, they are the set of real harmonic polynomials and the set of real Weyl tensors (see Subsection A.2). An interesting consequence of this is that the real representations carry an invariant quadratic form and we will compute here its signature, see Subsections A.1.1 and A.2.1.

A.1. The set of harmonic polynomials. From the Weyl dimension formula (see for example [23, Section 7.1.2]) it follows that the dimension of \mathcal{H}_p is

$$\dim \mathcal{H}_p = \binom{p+n-2}{p} \frac{2p+n-1}{n-1}.$$

The calculations for this are straightforward so we omit them. It suffices to plug $\lambda_1 = p$ and $\lambda_i = 0$ for all $i > 1$ in the Weyl dimension formulas for $SO(2l)$ and $SO(2l+1)$.

A.1.1. Invariant quadratic form. The aim of this section is to compute the signature $(n_+(p), n_-(p))$ of the $O_{\uparrow}(n, 1)$ -invariant quadratic form q on \mathcal{H}_p . General methods exist to compute this signature but this representation is simple enough so that a shorter derivation exists. See Subsection A.2.1 for a more robust method.

Note that the signature of the invariant quadratic form is not unique, but only up to swapping of $n_+(p)$ and $n_-(p)$ since a given invariant quadratic form can be multiplied by any non-zero number.

For simplicity we make the choice that $q(X^0) = -1$ and $q(X^i) = 1$ for all $i = 1, \dots, n$, and we extend q to $\mathbb{R}[X^0, X^1, \dots, X^n]$ by requiring that distinct monomials are orthogonal and that

$$q\left((X^0)^{k_0}(X^1)^{k_1} \dots (X^n)^{k_n}\right) = q(X^0)^{k_0} q(X^1)^{k_1} \dots q(X^n)^{k_n} = (-1)^{k_0}.$$

This form is the natural extension of the invariant quadratic form q on $(\mathbb{R}^{n,1})^*$ to $\text{Sym}_* \mathbb{R}^{n,1}$ and, as such, it is $O_{\uparrow}(n, 1)$ -invariant.

The signature $(n_{+,0}(p), n_{-,0}(p))$ of q restricted to $\mathbb{R}_p[X^0, X^1, \dots, X^n]$, the space of homogeneous polynomials of degree p , can be computed easily from the fact that monomials are orthogonal,

$$\begin{cases} n_{+,0}(p) = \#\{\text{monomials with even power of } X^0\}, \\ n_{-,0}(p) = \#\{\text{monomials with odd power of } X^0\}. \end{cases}$$

And thus,

$$\begin{cases} n_{+,0}(p) = \sum_{\substack{k \in \{0,1,\dots,p\} \\ k \text{ even}}} \binom{p-k+n-1}{n-1}, \\ n_{-,0}(p) = \sum_{\substack{k \in \{0,1,\dots,p\} \\ k \text{ odd}}} \binom{p-k+n-1}{n-1}. \end{cases}$$

This can be condensed into

$$\begin{cases} n_{+,0}(p) + n_{-,0}(p) = \sum_{k=0}^p \binom{p-k+n-1}{n-1}, \\ n_{+,0}(p) - n_{-,0}(p) = \sum_{k=0}^p (-1)^k \binom{p-k+n-1}{n-1}. \end{cases}$$

The space \mathcal{H}_p is the kernel of the map

$$\square : \mathbb{R}_p[X^0, X^1, \dots, X^n] \rightarrow \mathbb{R}_{p-2}[X^0, X^1, \dots, X^n]$$

and a complementary orthogonal subspace to it is given by the set \mathcal{K}_p of polynomials divisible by $-(X^0)^2 + (X^1)^2 + \dots + (X^n)^2$. It cannot be expected that \square is an isometry from \mathcal{K}_p to $\mathbb{R}_{p-2}[X^0, X^1, \dots, X^n]$. We have however the following result.

Lemma A.1. *The signature of q restricted to \mathcal{K}_p is the same as that on $\mathbb{R}_{p-2}[X^0, X^1, \dots, X^n]$.*

From this lemma it follows that the signature of the invariant quadratic form q on \mathcal{H}_k is given by

$$n_+(p) = n_{+,0}(p) - n_{+,0}(p-2), \quad n_-(p) = n_{-,0}(p) - n_{-,0}(p-2),$$

and thus

$$n_+(p) = \binom{p+n-1}{n-1}, \quad n_-(p) = \binom{p+n-2}{n-1}. \quad (55)$$

Proof of Lemma A.1. The proof is by a Brauer-style diagrammatic argument. The representation $\mathbb{R}_p[X^0, X^1, \dots, X^n]$ decomposes as a sum of irreducible subrepresentations,

$$\mathbb{R}_p[X^0, X^1, \dots, X^n] = \bigoplus_{k=0}^{\lfloor p/2 \rfloor} (-(X^0)^2 + (X^1)^2 + \dots + (X^n)^2)^k \mathcal{H}_{p-2k}.$$

It is not complicated to check that the map

$$\begin{aligned} m_k : \mathcal{H}_r &\rightarrow \mathbb{R}_{r+2k}[X^0, X^1, \dots, X^n] \\ P &\mapsto (-(X^0)^2 + (X^1)^2 + \dots + (X^n)^2)^k P \end{aligned}$$

preserves the quadratic form up to a positive constant, that is there exists a constant $\lambda_{k,r}$ such that for any $P \in \mathcal{H}_r$, we have $q(m_k(P)) = \lambda_{k,r}q(P)$.

We represent an arbitrary polynomial $Q \in \mathbb{R}_r[X^0, X^1, \dots, X^n]$ as a symmetric tensor with r indices,

$$Q = Q_{\mu_1 \dots \mu_r} X^{\mu_1} \dots X^{\mu_r}.$$

Multiplication by $(-(X^0)^2 + (X^1)^2 + \dots + (X^n)^2)^k$ is then the same as tensoring k times with $\eta = \eta_{\mu\nu} X^\mu X^\nu$ and symmeterizing. That is

$$\begin{aligned} (m_k(Q))_{\mu_1 \dots \mu_{r+2k}} &= Q_{(\mu_1 \dots \mu_r} \eta_{\mu_{r+1} \mu_{r+2}} \dots \eta_{\mu_{r+2k-1} \mu_{r+2k}}) \\ &= \frac{1}{(r+2k)!} \sum_{\sigma \in S_{r+2k}} Q_{\mu_{\sigma(1)} \dots \mu_{\sigma(r)}} \eta_{\mu_{\sigma(r+1)} \mu_{\sigma(r+2)}} \dots \eta_{\mu_{\sigma(r+2k-1)} \mu_{\sigma(r+2k)}}. \end{aligned}$$

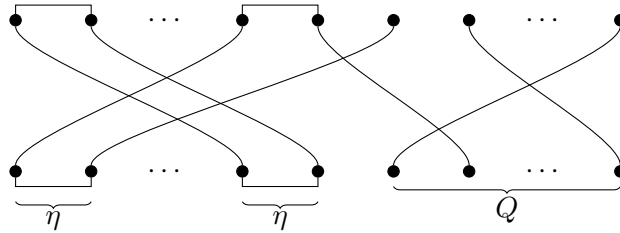
The form q can be written as follows:

$$q(Q) = Q_{\mu_1 \dots \mu_r} Q^{\mu_1 \dots \mu_r},$$

where indices are raised with respect to η . We have

$$\begin{aligned} q(m_k(Q)) &= \frac{1}{((r+2k)!)^2} \sum_{\sigma \in S_{r+2k}} Q_{\mu_{\sigma(1)} \dots \mu_{\sigma(r)}} \eta_{\mu_{\sigma(r+1)} \mu_{\sigma(r+2)}} \dots \eta_{\mu_{\sigma(r+2k-1)} \mu_{\sigma(r+2k)}} \\ &\quad \times \sum_{\tau \in S_{r+2k}} Q^{\mu_{\tau(1)} \dots \mu_{\tau(r)}} \eta^{\mu_{\tau(r+1)} \mu_{\tau(r+2)}} \dots \eta^{\mu_{\tau(r+2k-1)} \mu_{\tau(r+2k)}} \\ &= \frac{1}{(r+2k)!} Q^{\mu_1 \dots \mu_r} \eta^{\mu_{r+1} \mu_{r+2}} \dots \eta^{\mu_{r+2k-1} \mu_{r+2k}} \\ &\quad \times \sum_{\sigma \in S_{r+2k}} Q_{\mu_{\sigma(1)} \dots \mu_{\sigma(r)}} \eta_{\mu_{\sigma(r+1)} \mu_{\sigma(r+2)}} \dots \eta_{\mu_{\sigma(r+2k-1)} \mu_{\sigma(r+2k)}}. \end{aligned}$$

That is to say that $q(m_k(Q))$ is, up to a factor $\frac{1}{((r+2k)!)^2}$, the sum over all possible ways of pairing indices from two copies of $m_k(Q)$. Such a pairing can be represented as a Brauer-like diagram:



where there are k \curvearrowright and as many \curvearrowleft corresponding to the η 's we added to Q and $2r$ isolated points (r above and r below) corresponding to the r indices of Q . Each diagram contains (at most) three types of curves drawn by contractions (curvy lines) and η 's:

1. Closed curves (loops) corresponding to contractions of η 's with themselves so they give rise to a factor $(n+1)$ to the diagram.
2. Curves that connect the same Q (either the upper one or the lower one) corresponding to a contraction of Q with itself. Since Q is traceless, having such a curve in a diagram makes its contribution vanish.
3. Curves that connect the upper Q and the lower Q .

In the end, we get the following formula for $q(m_k(Q))$:

$$q(m_k(Q)) = \frac{q(Q)}{(r+2k)!} \sum_{D \in \mathcal{D}_0} (n+1)^{\#\text{loops}(D)},$$

where \mathcal{D}_0 denotes the set of diagrams for which there is no curve contracting the same Q . It follows that $q(m_k(Q)) = \lambda_{k,r} q(Q)$ with

$$\lambda_{k,r} = \frac{1}{(r+2k)!} \sum_{D \in \mathcal{D}_0} (n+1)^{\#\text{loops}(D)} > 0.$$

□

A.2. The set of polynomial Weyl tensors. Let \mathcal{E}_{p+2} be the set of solutions $h_{\mu\nu}$ of the Einstein equations linearized around the Minkowski metric η that are homogeneous polynomials of degree $p+2$. That is

$$0 = -\square h_{\mu\nu} + \partial_\mu \partial^\alpha h_{\alpha\nu} + \partial_\nu \partial^\alpha h_{\mu\alpha} - \partial_\mu \partial_\nu h^\alpha_\alpha - \partial^\alpha \partial^\beta h_{\alpha\beta} \eta_{\mu\nu} + \eta_{\mu\nu} \square h^\alpha_\alpha. \quad (56)$$

The following proposition gives an alternative characterization of the set of polynomial Weyl tensors \mathcal{W}_p .

Proposition A.2. *The set \mathcal{W}_p of polynomial Weyl tensors on $\mathbb{R}^{n,1}$ is the (dual of) the irreducible representation of $O_\uparrow(n,1)$ given by the following Young tableau*

$$Y_p = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline \end{array} \cdots \begin{array}{|c|} \hline p+4 \\ \hline \end{array}$$

This is the image of \mathcal{E}_{p+2} under the linearized Riemann tensor,

$$\mathcal{R} : h_{\mu\nu} \mapsto R_{\mu\nu\alpha\beta} := -\frac{1}{2} (\partial_\mu \partial_\alpha h_{\nu\beta} + \partial_\nu \partial_\beta h_{\mu\alpha} - \partial_\mu \partial_\beta h_{\nu\alpha} - \partial_\nu \partial_\alpha h_{\mu\beta}). \quad (57)$$

We start the proof by showing that given $W \in \mathcal{W}_p$ there exists a solution h to the linearised Einstein equations such that $W = \mathcal{R}(h)$.

Lemma A.3. *Given $W \in \mathcal{W}_p$ there exists $h \in \mathcal{E}_{p+2}$ such that $W = \mathcal{R}(h)$.*

The proof of this is based on a straight-forward refinement of the well-known Poincaré Lemma for cohomology, see for example [9, Chapter 1].

Lemma A.4. *If ω is a closed k -form ($k \geq 1$) on $\mathbb{R}^{n,1}$ with coefficients that are homogeneous polynomials of degree l , there exists a $(k-1)$ -form ξ whose coefficients are homogeneous polynomials of degree $l+1$ such that $\omega = d\xi$.*

Proof. The result is obvious for 0-forms. We assume now that $k > 0$. We introduce the operators $\mathcal{I}_k : \Lambda_k(\mathbb{R}^{n,1}) \rightarrow \Lambda_{k-1}(\mathbb{R}^{n,1})$, for all $k > 0$, defined as follows. For $\omega = f dX^{\mu_1} \wedge \cdots \wedge dX^{\mu_k} \in \Lambda_k(\mathbb{R}^{n,1})$ we set

$$\mathcal{I}_k(\omega)(p) := \sum_{j=1}^k (-1)^{j-1} X^{\mu_j} \int_0^1 t^{k-1} f(tp) dt dX^{\mu_1} \wedge \cdots \wedge \widehat{dX^{\mu_j}} \wedge \cdots \wedge dX^{\mu_k}.$$

and extend \mathcal{I}_k to all of $\Lambda_k(\mathbb{R}^{n,1})$ by linearity. We claim that for any $\omega \in \Lambda_k(\mathbb{R}^{n,1})$ we have

$$\omega = d(\mathcal{I}_k \omega) + \mathcal{I}_{k+1}(d\omega). \quad (58)$$

Indeed,

$$\begin{aligned} d(\mathcal{I}_k \omega)(p) &= \sum_{j=1}^k (-1)^{j-1} \int_0^1 t^{k-1} f(tp) dt dX^{\mu_j} \wedge dX^{\mu_1} \wedge \cdots \wedge \widehat{dX^{\mu_j}} \wedge \cdots \wedge dX^{\mu_k} \\ &\quad + \sum_{j=1}^k (-1)^{j-1} X^{\mu_j} \int_0^1 t^k \partial_\mu f(tp) dt dX^\mu \wedge dX^{\mu_1} \wedge \cdots \wedge \widehat{dX^{\mu_j}} \wedge \cdots \wedge dX^{\mu_k} \\ &= k \int_0^1 t^{k-1} f(tp) dt dX^{\mu_j} \wedge dX^{\mu_1} \wedge \cdots \wedge \widehat{dX^{\mu_j}} \wedge \cdots \wedge dX^{\mu_k} \\ &\quad + \sum_{j=1}^k (-1)^{j-1} X^{\mu_j} \int_0^1 t^k \partial_\mu f(tp) dt dX^\mu \wedge dX^{\mu_1} \wedge \cdots \wedge \widehat{dX^{\mu_j}} \wedge \cdots \wedge dX^{\mu_k}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{I}_{k+1}(d\omega)(p) &= \int_0^1 t^k X^\mu \partial_\mu f(tp) dt dX^{\mu_1} \wedge \cdots \wedge dX^{\mu_k} \\ &\quad - \sum_{j=1}^k (-1)^{j-1} X^{\mu_j} \int_0^1 t^k \partial_\mu f(tp) dt dX^\mu \wedge dX^{\mu_1} \wedge \cdots \wedge \widehat{dX^{\mu_j}} \wedge \cdots \wedge dX^{\mu_k}, \end{aligned}$$

so

$$\begin{aligned} d(\mathcal{I}_k \omega)(p) + \mathcal{I}_{k+1}(d\omega)(p) &= \int_0^1 \left(kt^{k-1} f(tp) + t^k X^\mu \partial_\mu f(tp) \right) dt dX^{\mu_1} \wedge \cdots \wedge dX^{\mu_k} \\ &= \int_0^1 \left(kt^{k-1} f(tp) + t^k \frac{d}{dt}(f(tp)) \right) dt dX^{\mu_1} \wedge \cdots \wedge dX^{\mu_k} \\ &= \left[t^k f(tp) \right]_0^1 dX^{\mu_1} \wedge \cdots \wedge dX^{\mu_k} \\ &= f(p) dX^{\mu_1} \wedge \cdots \wedge dX^{\mu_k}. \end{aligned}$$

In particular, if ω is closed, that is $d\omega = 0$, we have $\omega = d(\mathcal{I}\omega)$. It suffices to note that if ω has coefficients that are homogeneous polynomials of degree

l , then $\mathcal{I}\omega$ has coefficients that are homogeneous polynomials of degree $l + 1$. \square

We now prove Lemma A.3 in a sequence of claims.

Claim 1. There exists a tensor $f_{\mu\alpha\beta}$ which is antisymmetric with respect to the last two indices such that

$$W_{\mu\nu\alpha\beta} = \partial_\mu f_{\nu\alpha\beta} - \partial_\nu f_{\mu\alpha\beta}.$$

Further, the components of f are homogeneous polynomials of degree $p + 1$.

Proof. The second Bianchi identity for W can be rewritten

$$d(W_{\mu\nu\alpha\beta} dX^\mu \wedge dX^\nu) = 0.$$

Since $H^2(\mathbb{R}^{n,1}) = 0$, the existence of f follows from Lemma A.4. \square

Claim 2. We can assume that f satisfies

$$f_{\nu\alpha\beta} + f_{\alpha\beta\nu} + f_{\beta\nu\alpha} = 0.$$

Proof. Note that in the choice of f , we are free to add an arbitrary exact form $d\theta$. So we can change f to $\tilde{f} = f + d\theta$ where θ is an arbitrary $\Lambda_2(\mathbb{R}^{n,1})$ -valued 0-form. In component notation,

$$f_{\mu\alpha\beta} \rightsquigarrow \tilde{f}_{\mu\alpha\beta} = f_{\mu\alpha\beta} + \partial_\mu \theta_{\alpha\beta}.$$

The condition

$$\tilde{f}_{\nu\alpha\beta} + \tilde{f}_{\alpha\beta\nu} + \tilde{f}_{\beta\nu\alpha} = 0$$

reads

$$0 = f_{\nu\alpha\beta} + f_{\alpha\beta\nu} + f_{\beta\nu\alpha} + \partial_\nu \theta_{\alpha\beta} + \partial_\alpha \theta_{\beta\nu} + \partial_\beta \theta_{\nu\alpha}$$

or

$$0 = f_{\nu\alpha\beta} dX^\nu \wedge dX^\alpha \wedge dX^\beta + d(\theta_{\alpha\beta} dX^\alpha \wedge dX^\beta).$$

To guarantee the existence of a θ solving this equation, we have to prove that $f_{\nu\alpha\beta} dX^\nu \wedge dX^\alpha \wedge dX^\beta$ is closed. To do this, we use the first Bianchi identity for W ,

$$\begin{aligned} 0 &= W_{\mu\nu\alpha\beta} + W_{\alpha\mu\nu\beta} + W_{\nu\alpha\mu\beta} \\ &= \partial_\mu f_{\nu\alpha\beta} + \partial_\alpha f_{\mu\nu\beta} + \partial_\nu f_{\alpha\mu\beta} - \partial_\nu f_{\mu\alpha\beta} - \partial_\alpha f_{\nu\mu\beta} - \partial_\mu f_{\alpha\nu\beta} \\ &= \partial_\mu (f_{\nu\alpha\beta} - f_{\alpha\nu\beta}) + \partial_\nu (f_{\alpha\mu\beta} - f_{\mu\alpha\beta}) + \partial_\alpha (f_{\mu\nu\beta} - f_{\nu\mu\beta}), \end{aligned}$$

or

$$0 = d(f_{\mu\nu\beta} dX^\mu \wedge dX^\nu).$$

Taking the wedge product with dX^β , we obtain

$$d(f_{\mu\nu\beta} dX^\mu \wedge dX^\nu \wedge dX^\beta) = d(f_{\mu\nu\beta} dX^\mu \wedge dX^\nu) \wedge dX^\beta = 0$$

which is the desired identity. \square

Claim 3. There exists a tensor $h_{\alpha\beta}$ such that

$$f_{\mu\alpha\beta} = \partial_\alpha h_{\mu\beta} - \partial_\beta h_{\mu\alpha}.$$

Further the components of h are homogeneous polynomials of degree $p + 2$.

Proof. As for Claim 1, it suffices to see that $f_{\mu\alpha\beta}dX^\alpha \wedge dX^\beta$ is closed. This fact was already obtained in the proof of Claim 2. \square

Claim 4. The tensor $h_{\alpha\beta}$ can be assumed to be symmetric.

Proof. The solution $h_{\alpha\beta}$ from Claim 3 is not unique, the form $h_{\alpha\beta}dX^\beta$ is only defined up to the addition of an exact form dv_α . So we can change $h_{\alpha\beta}$ to $\tilde{h}_{\alpha\beta} = h_{\alpha\beta} + \partial_\beta v_\alpha$ without changing f (and hence W). We get a symmetric \tilde{h} if and only if the equation

$$0 = h_{\alpha\beta} - h_{\beta\alpha} + \partial_\beta v_\alpha - \partial_\alpha v_\beta$$

admits a solution v_α . This condition is the same as

$$d(v_\alpha dX^\alpha) = h_{\alpha\beta}dX^\alpha \wedge dX^\beta$$

so it suffices to check that $h_{\alpha\beta}dX^\alpha \wedge dX^\beta$ is closed. This follows from Claim 2. Indeed,

$$\begin{aligned} 0 &= f_{\nu\alpha\beta} + f_{\alpha\beta\nu} + f_{\beta\nu\alpha} \\ &= \partial_\alpha h_{\mu\beta} + \partial_\beta h_{\alpha\mu} + \partial_\mu h_{\beta\alpha} - \partial_\beta h_{\mu\alpha} - \partial_\alpha h_{\beta\mu} - \partial_\mu h_{\alpha\beta} \\ &= \partial_\mu(h_{\beta\alpha} - h_{\alpha\beta}) + \partial_\alpha(h_{\mu\beta} - h_{\beta\mu}) + \partial_\beta(h_{\alpha\mu} - h_{\mu\alpha}), \end{aligned}$$

so

$$0 = -i_{\partial_\mu} d(h_{\alpha\beta}dX^\alpha \wedge dX^\beta).$$

\square

Now W can be written in terms of h and we find that $W = -2\mathcal{R}(h)$. The condition (56) follows at once by the zero trace condition imposed on W since this condition is nothing but the vanishing of the linearized Ricci tensor around the Minkowski metric taken with respect to h .

We now want to identify a subrepresentation of the homogeneous polynomial solutions to the linearised Einstein equations with the property that it projects onto the set of Weyl tensors. A natural gauge condition in the context of gravitational waves is the so called de Donder's gauge condition,

$$h^\mu_\mu = 0, \quad \partial^\mu h_{\mu\nu} = 0. \quad (59)$$

The gauge freedom that appears for the linearized Einstein equations is the action of infinitesimal isometries,

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu,$$

for an arbitrary 1-form ξ_μ that is homogeneous polynomial of degree $p+3$. The main interest of this gauge condition is that Equation (56) reduces to the wave equation

$$\square h_{\mu\nu} = 0.$$

The fact that de Donder's gauge condition can be satisfied is the content of the next lemma.

Lemma A.5. *Given an arbitrary homogeneous polynomial symmetric 2-tensor $h_{\mu\nu}$ of degree $p+2$ solving (56), there exists an homogeneous polynomial 1-form ξ_μ of degree $p+3$ such that $\tilde{h}_{\mu\nu} := h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ satisfies de Donder's gauge condition (59). In particular, \tilde{h} satisfies the wave equation*

$$\square \tilde{h}_{\mu\nu} = 0.$$

Proof. Straightforward calculations show that de Donder's gauge condition for \tilde{h} imposes the following equations for ξ_μ ,

$$\partial^\mu \xi_\mu = -\frac{1}{2} h_\mu^\mu, \quad (60a)$$

$$\square \xi_\nu = -\partial^\mu h_{\mu\nu} + \frac{1}{2} \partial_\nu h_\mu^\mu. \quad (60b)$$

We remind the reader that the operator \square is a surjection from the set of homogeneous polynomials of degree $p+3$ to the set of homogeneous polynomials of degree $p+1$, so Equation (60b) admits a (highly non-unique) solution ξ_μ^0 . Upon changing $h_{\mu\nu}$ to $h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu^0 + \partial_\nu \xi_\mu^0$, we can assume that Equation (60b) reduces to

$$\square \xi_\nu = -\partial^\mu h_{\mu\nu} + \frac{1}{2} \partial_\nu h_\mu^\mu = 0. \quad (60b')$$

Taking the trace of Equation (56) for h together with Condition (60b'), we obtain that

$$\square h_\mu^\mu = 0.$$

As can be easily seen by taking the d'Alembertian of Equation (60a), this condition is necessary to find a ξ_μ solving (60a)-(60b). Conversely, since the space of harmonic homogeneous polynomials of degree $p+2$ is an irreducible representation of $O(n,1)$, it suffices to check that a single such polynomial $-\frac{1}{2} h_\mu^\mu$ can be written as $\partial^\mu \xi_\mu$. For example, if

$$-\frac{1}{2} h_\mu^\mu = (X^0 - X^1)^{p+2},$$

then we can choose

$$\xi = -\frac{1}{2(p+3)} (X^0 - X^1)^{p+3} (dX^0 + dX^1)$$

so that

$$\square \xi = 0 \text{ and } \partial^\mu \xi_\mu = -\frac{1}{2} h_\mu^\mu.$$

□

Note that de Donder's gauge condition does not determine ξ_μ completely, yet it serves an important purpose. Namely, any polynomial solution to the linearised Einstein equation has representatives modulo infinitesimal diffeomorphisms in the space $\mathcal{H}_{p+2} \otimes \text{Sym}_2(\mathbb{R}^{n,1})$, that is in the (dual of) the tensor product

$$\boxed{} \boxed{} \dots \boxed{} \otimes \boxed{} \boxed{}$$

The next step will be to decompose this tensor product into a sum of irreducible representations. The decomposition of the tensor product of two irreducible representations of $O_{\uparrow}(n, 1)$ is given by a generalization of the classical Littlewood-Richardson rules obtained in [35]. Yet, we need here more information. Then we will have to check which irreducible representation satisfies de Donder's gauge (this will ensure that the corresponding irreducible representation is a set of solutions to the Einstein equations) and does not belong to the kernel of the operator \mathcal{R} . It should be noted that checking either de Donder's gauge condition or the non-triviality of \mathcal{R} for an arbitrary irreducible representation can be done at the level of the highest weight vector. Indeed, both de Donder's gauge condition and \mathcal{R} can be viewed as intertwining maps for the action of $O_{\uparrow}(n, 1)$. And, under such a map, the image of an irreducible representation is either 0 or is isomorphic to the irreducible representation itself.

The decomposition of $\mathcal{H}_{p+2} \otimes \mathring{\text{Sym}}_2(\mathbb{R}^{n,1}) \otimes \mathbb{C}$ depends on whether $n = 3$ or $n \geq 3$.

Lemma A.6 (The case $n = 3$). *Assuming that $n = 3$, the representation $\mathcal{H}_{p+2} \otimes \mathring{\text{Sym}}_2(\mathbb{R}^{n,1})$ decomposes as the sum*

$$\begin{aligned} \mathcal{H}_{p+2} \otimes \mathring{\text{Sym}}_2(\mathbb{R}^{n,1}) \otimes \mathbb{C} = & V_{(p+4)\bar{\omega}_1 + (p+4)\bar{\omega}_2} \oplus V_{(p+4)\bar{\omega}_1 + (p+2)\bar{\omega}_2} \oplus V_{(p+2)\bar{\omega}_1 + (p+4)\bar{\omega}_2} \\ & \oplus V_{(p+4)\bar{\omega}_1 + p\bar{\omega}_2} \oplus V_{(p+2)\bar{\omega}_1 + (p+2)\bar{\omega}_2} \oplus V_{p\bar{\omega}_1 + (p+4)\bar{\omega}_2} \\ & \oplus V_{(p+2)\bar{\omega}_1 + p\bar{\omega}_2} \oplus V_{p\bar{\omega}_1 + (p+2)\bar{\omega}_2} \oplus V_{p\bar{\omega}_1 + p\bar{\omega}_2}, \end{aligned}$$

of irreducible $SO_{\uparrow}(n, 1)$ -representations. Here V_{ω} denotes the irreducible representation with highest weight ω . The first highest weight vectors are the following,

$$\begin{aligned} H_{(p+4)\bar{\omega}_1 + (p+4)\bar{\omega}_2} &= (X^0 + X^1)^{p+2} (dX^0 + dX^1)^2, \\ H_{(p+2)\bar{\omega}_1 + (p+4)\bar{\omega}_2} &= (X^0 + X^1)^{p+1} (X^2 + iX^3) (dX^0 + dX^1)^2 \\ &\quad - (X^0 + X^1)^{p+2} (dX^0 + dX^1) (dX^2 + idX^3), \\ H_{(p+4)\bar{\omega}_1 + (p+2)\bar{\omega}_2} &= (X^0 + X^1)^{p+1} (X^2 - iX^3) (dX^0 + dX^1)^2 \\ &\quad - (X^0 + X^1)^{p+2} (dX^0 + dX^1) (dX^2 - idX^3), \\ H_{p\bar{\omega}_1 + (p+4)\bar{\omega}_2} &= (X^0 + X^1)^p [(X^2 + iX^3) (dX^0 + dX^1) - (X^2 + iX^3) (dX^2 + idX^3)]^2, \\ H_{(p+4)\bar{\omega}_1 + p\bar{\omega}_2} &= (X^0 + X^1)^p [(X^2 - iX^3) (dX^0 + dX^1) - (X^2 - iX^3) (dX^2 + idX^3)]^2, \\ h_{(p+2)\bar{\omega}_1 + (p+2)\bar{\omega}_2} &= \frac{p+2}{2} (X^0 + X^1)^{p+2} ((dX^0)^2 - (dX^1)^2 + (dX^2)^2 + (dX^3)^2) \\ &\quad - 2(p+2)(X^0 + X^1)^{p+1} \\ &\quad - 2(p+2)(X^0 + X^1)^2 (X^2 + X^3) (dX^0 + dX^1) (dX^2 + dX^3) \\ &\quad + [(p+1)((X^2)^2 + (X^3)^2) + (X^0)^2 - (X^1)^2] (X^0 + X^1)^p (dX^0 + dX^1)^2. \end{aligned} \tag{61}$$

It should be noted that we only give the formulas for the first highest weight vectors. The last three highest weight vectors can be computed but their expression are rather long and uninstrusive. The proof of Lemma A.6 is a simple exercise of the computation of the Clebsch-Gordan coefficients. For more details on this method, the interested reader can consult reference textbook on quantum mechanics such as [15, Chapter X]. As an example, the vectors $H_{(p+2)\bar{\omega}_1+(p+4)\bar{\omega}_2}$ and $H_{p\bar{\omega}_1+(p+4)\bar{\omega}_2}$ can be computed to be

$$\begin{aligned} H_{(p+2)\bar{\omega}_1+(p+4)\bar{\omega}_2} &= \left(f_1 \otimes 1 - \frac{p+2}{2} 1 \otimes f_1 \right) H_{(p+4)\bar{\omega}_1+(p+4)\bar{\omega}_2} \\ H_{p\bar{\omega}_1+(p+4)\bar{\omega}_2} &= \left(\frac{1}{p+1} f_1^2 \otimes 1 - f_1 \otimes f_1 + \frac{p+2}{2} 1 \otimes f_1^2 \right) H_{(p+4)\bar{\omega}_1+(p+4)\bar{\omega}_2}. \end{aligned}$$

Lemma A.7 (The case $n > 3$). *Assuming that $n > 3$, the representation $\mathcal{H}_{p+2} \otimes \mathring{\text{Sym}}_2(\mathbb{R}^{n,1})$ decomposes as the sum*

$$\begin{aligned} \mathcal{H}_{p+2} \otimes \mathring{\text{Sym}}_2(\mathbb{R}^{n,1}) \otimes \mathbb{C} &= V_{(p+4)\bar{\omega}_1} \oplus V_{(p+2)\bar{\omega}_1+\bar{\omega}_2} \oplus V_{p\bar{\omega}_1+2\bar{\omega}_2} \\ &\quad \oplus V_{(p+2)\bar{\omega}_1} \oplus V_{p\bar{\omega}_1+\bar{\omega}_2} \\ &\quad \oplus V_{p\bar{\omega}_1}, \end{aligned}$$

of irreducible $SO_{\uparrow}(n,1)$ -representations. Here V_{ω} denotes the irreducible representation with highest weight ω . The highest weight vectors are the

following,

$$\begin{aligned}
H_{(p+4)\bar{\omega}_1} &= (Z^{-1})^{p+2} dZ^{-1} \otimes dZ^{-1}, \\
H_{(p+2)\bar{\omega}_1 + \bar{\omega}_2} &= (Z^{-1})^{p+1} Z^{-2} dZ^{-1} \otimes dZ^{-1} - \frac{1}{2} (Z^{-1})^{p+2} (dZ^{-1} \otimes dZ^{-2} + dZ^{-2} \otimes dZ^{-1}), \\
H_{p\bar{\omega}_1 + 2\bar{\omega}_2} &= (Z^{-1})^{p+2} dZ^{-2} \otimes dZ^{-2} - Z^{-2} (Z^{-1})^{p+1} (dZ^{-1} \otimes dZ^{-2} + dZ^{-2} \otimes dZ^{-1}) \\
&\quad + (Z^{-1})^p (Z^{-2})^2 dZ^{-1} \otimes dZ^{-1}, \\
h_{(p+2)\bar{\omega}_1} &= \frac{1}{2} (Z^{-1})^{p+1} \sum_j Z^j (dZ^{-1} \otimes dZ^j + dZ^j \otimes dZ^{-1}) \\
&\quad - \frac{p+1}{2p+n+1} (Z^{-1})^p \sum_j Z^j Z^{-j} dZ^{-1} \otimes dZ^{-1} \\
&\quad - \frac{1}{n+1} (Z^{-1})^{p+2} \sum_j dZ^j \otimes dZ^{-j}, \\
h_{p\bar{\omega}_1 + \bar{\omega}_2} &= \frac{1}{2} \sum_j (Z^{-1})^{p+1} Z^j (dZ^{-2} \otimes dZ^{-j} + dZ^{-j} \otimes dZ^{-2}) \\
&\quad - \frac{1}{2} \sum_j (Z^{-1})^p Z^{-2} Z^j (dZ^{-1} \otimes dZ^{-j} + dZ^{-j} \otimes dZ^{-1}) \\
&\quad - \frac{p}{2(2p+n+1)} \left(\sum_j Z^j Z^{-j} \right) \times \\
&\quad \quad \left[(Z^{-1})^p (dZ^{-1} \otimes dZ^{-2} + dZ^{-2} \otimes dZ^{-1}) - 2(Z^{-1})^{p-1} Z^{-2} dZ^{-1} \otimes dZ^{-1} \right], \\
k_{p\bar{\omega}_1} &= (Z^{-1})^p \left(\sum_j Z^j dZ^{-j} \right)^{\otimes 2} \\
&\quad - \frac{1}{2p+n+1} \left(p(Z^{-1})^{p-1} \sum_j Z^j (dZ^{-1} \otimes dZ^{-j} + dZ^{-j} \otimes dZ^{-1}) + 2(Z^{-1})^p \sum_j dZ^j \otimes dZ^{-j} \right) \\
&\quad + \frac{p(p-1)}{8(2p+n+1)(2p+n-1)} (Z^{-1})^{p-2} \left(\sum_j Z^j Z^{-j} \right)^2 dZ^{-1} \otimes dZ^{-1}.
\end{aligned} \tag{62}$$

Here Z^i denotes coordinates on $\mathbb{R}^{n,1}$ associated to the basis $e_{\pm j}$.

Proof. The proof of this lemma appears to be pretty tedious, yet fairly straightforward. Therefore, we give only the strategy. From [35], we know

that

$$\begin{array}{c}
 \overbrace{\boxed{} \cdots \boxed{}}^{p \text{ boxes}} \otimes \boxed{} \boxed{} = \overbrace{\boxed{} \cdots \boxed{}}^{p+4 \text{ boxes}} \oplus \overbrace{\begin{array}{c} \boxed{} \cdots \boxed{} \\ \boxed{} \end{array}}^{p+3 \text{ boxes}} \oplus \overbrace{\begin{array}{c} \boxed{} \cdots \boxed{} \\ \boxed{} \end{array}}^{p+2 \text{ boxes}} \\
 \oplus \overbrace{\boxed{} \cdots \boxed{}}^{p+2 \text{ boxes}} \oplus \overbrace{\begin{array}{c} \boxed{} \cdots \boxed{} \\ \boxed{} \end{array}}^{p+1 \text{ boxes}} \oplus \overbrace{\boxed{} \cdots \boxed{}}^{p \text{ boxes}}
 \end{array} \tag{63}$$

This shows that the highest weight vectors in the tensor product $\mathcal{H}_{p+2} \otimes \mathring{\text{Sym}}_2(\mathbb{R}^{n,1}) \otimes \mathbb{C}$ have the highest weights indicated in the statement of the lemma. The first three irreducible representations that appear in the right hand side of Equation (63) have $p+4$ boxes so the corresponding highest weight vector in $\mathcal{H}_{p+2} \otimes \mathring{\text{Sym}}_2(\mathbb{R}^{n,1})$ can be obtained by using the corresponding Young symmetrizer, see [23, Lemma 9.3.2]. For the fourth and the fifth terms in Equation (63), the same procedure leads to elements in $\mathcal{H}_{p+1} \otimes (\mathbb{R}^{n,1})^* \otimes \mathbb{C}$. One then has to multiply them by the “metric” $\sum_j Z^j dZ^{-j}$. However, these elements do not belong to $\mathcal{H}_{p+2} \otimes \mathring{\text{Sym}}_2(\mathbb{R}^{n,1}) \otimes \mathbb{C}$ since neither its d’Alembertian nor its trace are zero so we need to subtract them by adding terms that are proportional to $\sum_j Z^j Z^{-j}$ and to $\sum_j dZ^j \otimes dZ^{-j}$. All these maps being intertwining for $O(n+1, \mathbb{C})$, we have then constructed the highest weight vector of the corresponding representation. The procedure is similar for the sixth term in (63). However, here, we have to multiply by $\left(\sum_j Z^j dZ^{-j}\right)^{\otimes 2}$. \square

The last step in the proof of Proposition A.2 is to identify \mathcal{W}_p with the irreducible representation of highest weight $p\bar{\omega}_1 + 2\bar{\omega}_2$ if $n > 3$ or to $V_{p\bar{\omega}_1 + (p+4)\bar{\omega}_2} \oplus V_{(p+4)\bar{\omega}_1 + p\bar{\omega}_2}$ if $n = 3$.

From Lemma A.5, we know that any $W \in \mathcal{W}_p$ can be represented as $W = \mathcal{R}(h)$ for some $h \in \mathcal{E}_{p+2}$ satisfying de Donder’s gauge condition. In particular, h belongs to $\mathcal{H}_{p+2} \otimes \mathring{\text{Sym}}_2(\mathbb{R}^{n,1})$. This means that \mathcal{W}_p is the image of the set of $O_\uparrow(n, 1)$ irreducible subrepresentations of $\mathcal{H}_{p+2} \otimes \mathring{\text{Sym}}_2(\mathbb{R}^{n,1})$ that satisfy de Donder’s gauge condition. It can be checked that only the first three vectors in (62), which we denoted H_ω to distinguish them from the others, and similarly only the first five vectors in (61) satisfy this condition (as mentioned earlier, this gauge condition must only be verified for the highest weight vector).

We now restrict to $n > 3$, the case $n = 3$ being similar. The first two terms actually belong to the kernel of \mathcal{R} . The simplest way to check this is

by noticing that

$$\begin{aligned} H_{(p+4)\overline{\omega}_1} &= \mathcal{L}_{\xi_{(p+2)\overline{\omega}_1}} \eta, \\ H_{(p+2)\overline{\omega}_1 + \overline{\omega}_2} &= \mathcal{L}_{\xi_{(p+2)\overline{\omega}_1 + \overline{\omega}_2}} \eta, \end{aligned}$$

where

$$\begin{aligned} \xi_{(p+4)\overline{\omega}_1} &= \frac{1}{2(p+3)} (Z^{-1})^{p+3} e_{+1}, \\ \xi_{(p+2)\overline{\omega}_1 + \overline{\omega}_2} &= \frac{1}{2(p+2)} ((Z^{-1})^{p+2} Z^{-2} e_1 - (Z^{-1})^{p+3} e_{+2}). \end{aligned} \quad (64)$$

So only the third term in (63) remains and it is simple matter to check that

$$\mathcal{R}(H_{p\overline{\omega}_1 + 2\overline{\omega}_2})(e_{-1}, e_{-2}, e_{-1}, e_{-2}) = (p+2)(p+3)(Z^{-1})^p \neq 0.$$

This proves that \mathcal{W}_p is the image of $V_{p\overline{\omega}_1 + 2\overline{\omega}_2}$ under \mathcal{R} , and thus completing the proof of Proposition A.2.

The dimension of \mathcal{W}_p is

$$\dim \mathcal{W}_p = \frac{1}{2} \frac{n+1}{n-1} \binom{p+n}{p+3} \frac{(p+1)(p+n+2)(2p+n+3)}{p+n}.$$

This formula follows once again by the Weyl dimension formula. Note that the case $n+1=4$ has to be checked separately since the representation $\mathcal{W}_p = (p, p+4) \oplus (p+4, p)$ is not irreducible for $n+1=4$.

A.2.1. Invariant quadratic form. To compute the signature of the invariant quadratic form on \mathcal{W}_p , we use the Cartan decomposition of $O_{\uparrow}(n, 1)$ (see for example [30] for a presentation of the Cartan decomposition). The maximal compact subgroup K of $O_{\uparrow}(n, 1)$ is $O(n)$, where it is understood that elements in $O(n)$ leave invariant the time direction X^0 . Let \mathfrak{p} denote the subspace of the Lie algebra $\mathfrak{so}(n, 1)$ generated by the boosts a_i , $i = 1, \dots, n$.

The subspace \mathfrak{p} is left invariant by the adjoint action of $O(n)$. As such, \mathfrak{p} is a representation of $O(n)$ which turns out to be equivalent to the standard one.

From the classical branching rules (see for example [23, Chapter 8]), we know how \mathcal{W}_p decomposes into representations of $O(n)$,

$$\begin{aligned} \text{Res}_{O(n)}^{O_{\uparrow}(n,1)}(\mathcal{W}_p) &= \overbrace{\begin{array}{|c|c|c|c|} \hline \square & \square & \cdots & \square \\ \hline \square & \square & & \square \\ \hline \end{array}}^{p+2 \text{ boxes}} \oplus \cdots \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \\ &\oplus \overbrace{\begin{array}{|c|c|c|c|} \hline \square & \square & \cdots & \square \\ \hline \square & & & \square \\ \hline \end{array}}^{p+2 \text{ boxes}} \oplus \cdots \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \\ &\oplus \overbrace{\begin{array}{|c|c|c|c|} \hline \square & \cdots & \square & \square \\ \hline \end{array}}^{p+2 \text{ boxes}} \oplus \cdots \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \end{aligned} \quad (65)$$

An important fact to notice is that the restriction is multiplicity free and this decomposition holds true considering either the real representation \mathcal{W}_p or the complex representation $\mathcal{W}_p \otimes \mathbb{C}$.

We now study the action of \mathfrak{p} closer. The mapping $a \otimes W \mapsto a \cdot W$ from $\mathfrak{p} \otimes \mathcal{W}_p$ to \mathcal{W}_p is $O(n)$ -equivariant meaning that for any $R \in O(n)$, we have $Ad(R)(a) \cdot (R_*W) = R_*(a \cdot W)$ (we remind the reader that $O_{\uparrow}(n, 1)$ acts on \mathcal{W}_p by push-forward). As a consequence, elements belonging to a irreducible representation V_0 of $O(n)$ in the righthand side of (65) are mapped by \mathfrak{p} into the subspace of $\text{Res}_{O(n)}^{O_{\uparrow}(n, 1)}$ which is the direct sum of irreducible representations that appear both in the righthand side of (65) and in the decomposition of the tensor product $V_0 \otimes \mathbb{R}^n$ into irreducible representations of $O(n)$. As an example,

$$\mathfrak{p} \text{ maps } \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \text{ to } \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}.$$

What is important to notice is that $O(n)$ -representations with an odd number of boxes are mapped to representations with an even number of boxes and vice versa by \mathfrak{p} .

The invariant quadratic form q on \mathcal{W}_p restricts to each element in the righthand side of Equation (65). Being real representation of $O(n)$, each of them carry a unique (up to normalization) $O(n)$ -invariant quadratic form which can be chosen to be positive definite. We conclude that the restriction of q to any element in the righthand side of (65) is either positive or negative definite.

The Lie group with Lie algebra $\mathfrak{g} = \mathfrak{o}(n) \oplus i\mathfrak{p}$ is $SO(n+1)$, the compact form of $SO_{\uparrow}(n, 1)$. The representation $\mathcal{W}_p \otimes \mathbb{C}$ is also of real type for $SO(n+1)$. A real form of $\mathcal{W}_p \otimes \mathbb{C}$ for $SO(n+1)$ can be constructed as follows. Let us abuse notation and assume that a Young diagram represents the real representation inside \mathcal{W}_p that is isomorphic to the image of the Young projector.

Then a real form for $SO(n+1)$ is given by

$$\widetilde{\mathcal{W}}_p = \left[\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \oplus \dots \right] \oplus i \left[\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \oplus \dots \right],$$

namely, we put an i in front of all subrepresentations with an odd number of boxes. Since $SO(n+1)$ is a compact Lie group, $\widetilde{\mathcal{W}}_p$ carries a unique invariant quadratic form q which is positive definite.

Looking at the invariant quadratic form q on \mathcal{W}_p this means that q is positive definite on the $O(n)$ -irreducible representations with an even number of boxes and negative definite on the representations with an odd number of boxes.

If we let $(n_+(p), n_-(p))$ denote the signature of q on \mathcal{W}_p , we have $n_+(p) + n_-(p) = \dim \mathcal{W}_p$ and $n_+(p) - n_-(p)$ can be computed by induction on p .

Indeed,

$$n_+(p) - n_-(p) = n_+(p-1) - n_-(p-1)$$

$$+ (-1)^p \left[\dim \overbrace{\begin{array}{|c|c|c|c|} \hline \square & \cdots & \square & \square \\ \hline \end{array}}^{p+2 \text{ boxes}} - \dim \overbrace{\begin{array}{|c|c|c|c|} \hline \square & \square & \cdots & \square \\ \hline \square & & & \end{array}}^{p+2 \text{ boxes}} + \dim \overbrace{\begin{array}{|c|c|c|c|} \hline \square & \square & \cdots & \square \\ \hline \square & \square & & \end{array}}^{p+2 \text{ boxes}} \right].$$

For $p = 0$, we find

$$n_+(0) - n_-(0) = \frac{1}{12}(n+2)(n-1)(n-2)(n-3).$$

The following formula can be obtained by induction,

$$n_+(p) - n_-(p) = (-1)^p \frac{p+1}{2} (p+n+2) \binom{p+n-1}{p+3}.$$

At this point it should be noted that it is much more convenient to change the convention and assume that q is positive definite if p is even and negative definite if p is odd. That has the effect of removing the annoying term $(-1)^p$. With this new convention, we get

$$\begin{cases} n_+(p) = \frac{1}{2}(n^2 + (n+1)p + 3) \frac{(p+1)(p+n+2)}{(n-1)(p+n)} \binom{p+n}{p+3}, \\ n_-(p) = \frac{1}{2}(np + 4n + p) \frac{(p+1)(p+n+2)}{(n-1)(p+n)} \binom{p+n}{p+3}. \end{cases}$$

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